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Extended Subloading-overstress-Gurson Model with Ductile Damage

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The Gurson model is capable of describing the ductile damage of metals. It is extended to describe the cyclic viscoplastic deformation behavior by incorporating the subloading-overstress model in this article. It is called the *extended subloading-overstress-Gurson model*.

Key Words: *Cyclic loading, Ductile damage, Viscolastoplastic deformation, Gurson model, Metals, Subloading-overstress model. Isotropic hardening stagnation*

1. INTRODUCTION

The ductile damage of metals is caused by the growth of the voids (or cavities). The plastic deformation with the ductile damage is influenced by the hydrostatic stress even if the base material is the Mises material, the plastic deformation behavior of which is independent of mean stress. The elastoplastic constitutive model taken account of the existence of voids was proposed first by Gurson [1]. It is often called the *Gurson model*. Further, it has been extended to describe the influence of the nucleation and the growth of voids by Needleman and Rice [2], Chu and Needleman [3], Tvergaard and Needleman [4], Needleman and Tvergaard [5], etc. The extended model is called the *GTN (Gurson-Tvergaard-Needleman) model*. It will be extended to describe the cyclic viscoplastic deformation by incorporating the extended subloading-overstress model (Hashiguchi [6]; Hashiguchi et al. [7][8]) with the isotropic hardening stagnation formulation in APPENDIX in this article. The extended model is called the *subloading-overstress Gurson model*.

2. STRAIN AND ITS DECOMPISITION INTO ELASTIC AND VISCOPLASTIC PARTS

The strain $\boldsymbol{\varepsilon}$ is additively decomposed into the elastic strain $\boldsymbol{\varepsilon}^e$ and the viscoplastic strain $\boldsymbol{\varepsilon}^{vp}$ as follows:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp} \tag{1}$$

the time-differentiation of which leads to

$$\overset{\bullet}{\boldsymbol{\varepsilon}} = \overset{\bullet}{\boldsymbol{\varepsilon}}{}^e + \overset{\bullet}{\boldsymbol{\varepsilon}}{}^{vp} \tag{2}$$

3. ELASTIC CONSTITUTIVE EQUATION

Let the Cauchy stress $\boldsymbol{\sigma}$ be given by the elastic strain $\boldsymbol{\varepsilon}^e$ as follows:

$$\boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\varepsilon}^e, \quad \boldsymbol{\varepsilon}^e = \mathbb{E}^{-1}\boldsymbol{\sigma} \tag{3}$$

which leads to the following rate relation by the time-differentiation, where $\mathbb{E}$ is the fourth-order elastic modulus tensor. Equation (3) leads to

$$\overset{\bullet}{\boldsymbol{\sigma}} = \mathbb{E} : \overset{\bullet}{\boldsymbol{\varepsilon}}{}^e, \quad \overset{\bullet}{\boldsymbol{\varepsilon}}{}^e = \mathbb{E}^{-1}\overset{\bullet}{\boldsymbol{\sigma}} \tag{4}$$

$$\overset{\bullet}{\boldsymbol{\sigma}} = \mathbb{E} : (\overset{\bullet}{\boldsymbol{\varepsilon}} - \overset{\bullet}{\boldsymbol{\varepsilon}}{}^{vp}) \tag{5}$$

under the assumption that $\mathbb{E}$ is the constant tensor. Here, let $\mathbb{E}$ be given by Hooke's law:

$$\left.\begin{aligned}
&\mathbb{E} = \frac{E}{1+\nu}\Big(\mathbb{S} + \frac{\nu}{1-2\nu}\mathbb{T}\Big),\\
&\mathbb{E}_{ijkl} = \frac{E}{1+\nu}\Big[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\Big]\\
&\mathbb{E}^{-1} = \frac{1}{E}\big[(1+\nu)\mathbb{S} - \nu\mathbb{T}\big],\\
&\mathbb{E}^{-1}_{ijkl} = \frac{1}{E}\Big[\frac{1}{2}(1+\nu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \nu\delta_{ij}\delta_{kl}\Big]
\end{aligned}\right\} \tag{6}$$

where E is Young's modulus and ν is Poisson's ratio, and

$$\mathbb{S} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \ \mathbb{T} = \delta_{ij}\delta_{kl} \tag{7}$$

4. GURSON'S YIELD SURFACE

The following yield condition was derived by Gurson [1] based on the axisymmetric rigid-plastic deformation analysis of the rigid-perfectly plastic Mises metal containing a concentric spherical cavity.

$$f(\boldsymbol{\sigma}, F, \xi) = \Big(\frac{\sigma^{eq}}{F}\Big)^2 + 2\xi\cosh\Big(\frac{3}{2}\frac{\sigma_m}{F}\Big) - \xi^2 - 1 = 0 \tag{8}$$

where

$$\sigma_m \equiv (1/3)\mathrm{tr}\boldsymbol{\sigma}, \ \sigma^{eq} = \sqrt{3/2}\,\|\boldsymbol{\sigma}'\| \tag{9}$$

$\boldsymbol{\sigma}$ is the Cauchy stress, $(\ )'$ designating the deviatoric part.

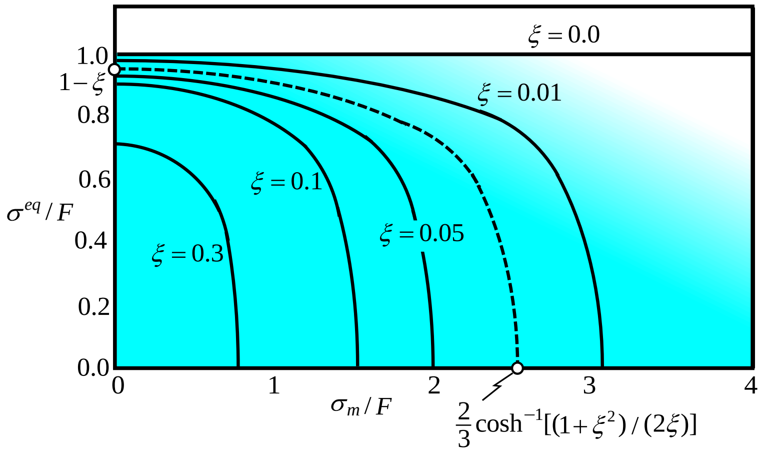

Fig. 1 Influence of void volume fraction in Gurson's yield surface which is illustrated only in the first orthant because of the point-symmetry.

F is the hardening function and ξ is the damage variable defined by the void volume fraction, i.e. the volume of the base material per unit aggregate volume. Equation (8) is reduced to the von Mises yield condition, i.e. $\sigma^{eq} = F$ for $\xi = 0$ in the undamaged state. The dependence of the yield surface on the pressure in Eq. (8) is shown in **Fig. 1**, which is the point-symmetry in the stress space.

5. EXTENDED SUBLOADING-GURSON YIELD AND SUBLOADING SURFACES

The above-mentioned Gurson model will be extended to describe the viscoplastic deformation induced by the rate of stress which does not lie on the yield surface in general by incorporating the subloading surface model (Hashiguchi [6], Hashiguchi et al. [7][8]) in this section.

The *subloading surface*, which passes through the current stress point $\bar{\boldsymbol{\sigma}}$ and is the similar to the yield surface in Eq. (8) with respect to the elastic-core $\mathbf{c}$, is described by the following equation (see **Fig. 2**):

$$f(\bar{\boldsymbol{\sigma}}, RF, \xi) = \left(\frac{\bar{\sigma}^{eq}}{RF}\right)^2 + 2\xi\cosh\left(\frac{3}{2}\frac{\bar{\sigma}_m}{RF}\right) - \xi^2 - 1 = 0 \quad (10)$$

where R is the *yield ratio*, i.e. the size of the subloading surface to the yield surface and calculated from this nonlinear equation and

$$\bar{\sigma}_m \equiv (1/3)\mathrm{tr}\bar{\boldsymbol{\sigma}},\ \bar{\boldsymbol{\sigma}}' \equiv \bar{\boldsymbol{\sigma}} - \bar{\sigma}_m/3,\ \bar{\sigma}^{eq} \equiv \sqrt{3/2}\,\|\bar{\boldsymbol{\sigma}}'\| \quad (11)$$

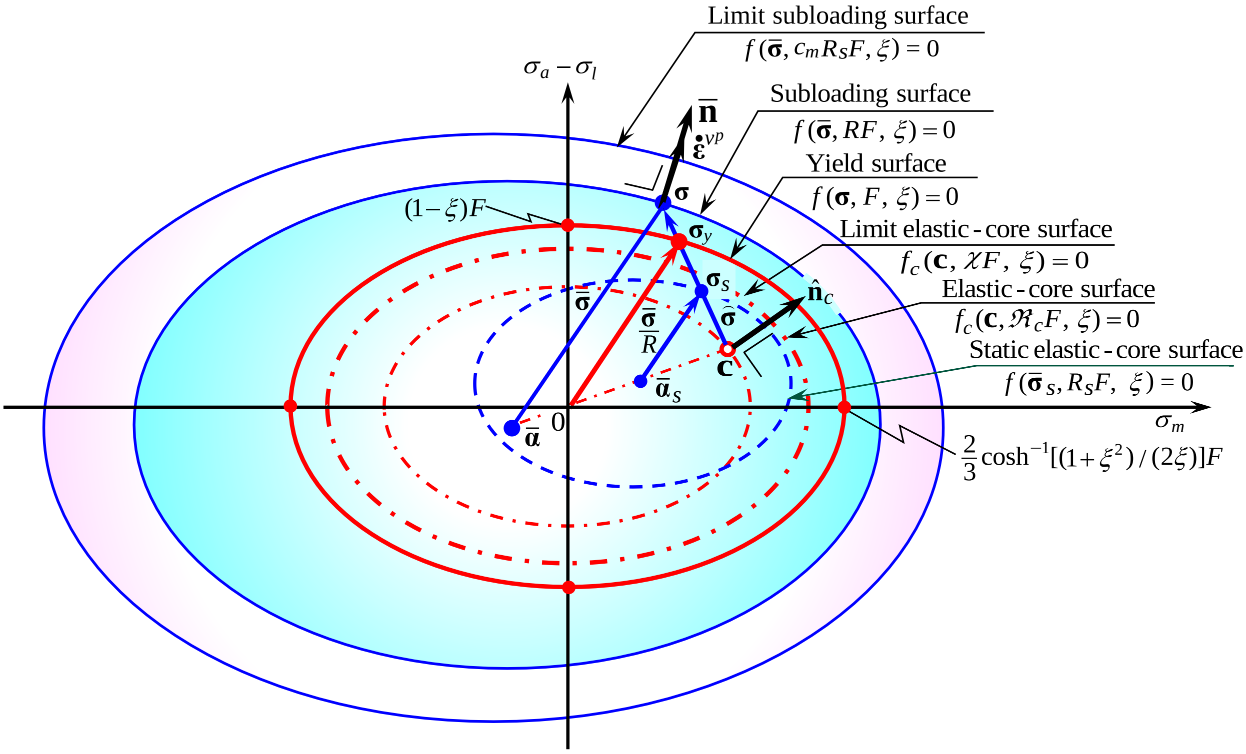

Fig. 2. Limit subloading, subloading, limit elastic-core, elastic-core and static subloading surfaces in two-dimensional stress space $(\sigma_m, \sigma_a - \sigma_l)$ in axisymmetric stress loading state, where σ_a and σ_l are axial stress and lateral stress, respectively.

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}} = \boldsymbol{\sigma} - (1+R)\mathbf{c} \quad (12)$$

$$\bar{\boldsymbol{\alpha}} = (1+R)\mathbf{c} \quad (13)$$

$\bar{\boldsymbol{\alpha}}$ stands for the conjugate (similar) point in the subloading surface to the center of the yield surface, i.e. the origin of the stress space. $\mathbf{c}$ is the similarity-center of the yield and the subloading surfaces, which is called the *elastic-core*, since the purely-elastic deformation behavior is induced when the stress lies on the point $\mathbf{c}$ leading to $R = 0$.

The evolution rule of the elastic-core $\mathbf{c}$ will be formulated in the later section 9.

6. EVOLUTION OF DAMAGE VARIABLE

The rate of the damage variable, i,e, the void volume fraction $\dot{\xi}$ is given by the sum of the growth rate $\dot{\xi}_{grow}$ and the nucleation rate of new voids $\dot{\xi}_{nucl}$ as follows:

$$\dot{\xi} = \dot{\xi}_{grow} + \dot{\xi}_{nucl} \quad (14)$$

In what follows, the damage variable will be formulated by modifying the plastic strain rate to the viscoplastic strain rate.

First, the rate of the void growth would be induced by the rate of volumetric strain of the whole aggregate, since the base material is assumed to be plastically incompressible, and thus it is given as follows (Needleman and Rice [2]):

$$\dot{\xi}_{grow} = (1-\xi)\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}^{vp} \quad (15)$$

Further, the nucleation of void would be caused by the magnitude of the viscoplastic strain rate and the decrease of the positive pressure, and thus it is given as follows (Chu and Needleman [3]):

$$\dot{\xi}_{nucl} = a_1 \|\dot{\boldsymbol{\varepsilon}}^{vp}\| + a_2 \langle \mathrm{tr}\,\dot{\boldsymbol{\sigma}} \rangle / 3 \quad (16)$$

where the coefficients a_1 and a_2 are the material constants and $\langle\ \rangle$ is Macauley bracket, i.e. $\langle s \rangle = s$ for $s > 0$ and $\langle s \rangle = 0$ for $s \le 0$ with the scalar variable s. The substitution of Eqs. (15) and (16) into Eq. (14) leads to

$$\dot{\xi} = (1-\xi)\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}^{vp} + a_1 \|\dot{\boldsymbol{\varepsilon}}^{vp}\| + a_2 \langle \dot{\sigma}_m \rangle$$

$$= (1-\xi)\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}^{vp} + a_1 \|\dot{\boldsymbol{\varepsilon}}^{vp}\| + a_2 \langle \mathrm{tr}\,\dot{\boldsymbol{\sigma}} \rangle / 3 \quad (17)$$

7. VISCOPLASTIC STRIN RATE

The viscoplastic strain rate is induced by the overstress from the *static subloading surface* (see **Fig. 2**) expressed in the following equation which is given by replacing the current stress $\boldsymbol{\sigma}$, the center $\bar{\boldsymbol{\alpha}}$ and the yield ratio R to their conjugate points $\boldsymbol{\sigma}_s$, $\bar{\boldsymbol{\alpha}}_s$ and the *static yield ratio* R_s (≤ 1), respectively, in the subloading surface in Eq. (10).

$$f(\bar{\boldsymbol{\sigma}}_s,\ R_s F,\ \xi)$$

$$= \left(\frac{\bar{\sigma}_s^{eq}}{R_s F}\right)^2 + 2\xi \cosh\left(\frac{3}{2}\frac{\bar{\sigma}_{sm}}{R_s F}\right) - \xi^2 - 1 = 0 \quad (18)$$

with

$$\bar{\boldsymbol{\sigma}}_s = \boldsymbol{\sigma}_s - \bar{\boldsymbol{\alpha}}_s = \frac{R_s}{R}\bar{\boldsymbol{\sigma}} \quad (19)$$

$$\left(\frac{\boldsymbol{\sigma}_s - \bar{\boldsymbol{\alpha}}_s}{R_s} = \frac{\boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}}}{R},\ \boldsymbol{\sigma}_s - \mathbf{c} = R_s(\boldsymbol{\sigma}_y - \mathbf{c}) = R_s\frac{\boldsymbol{\sigma} - \mathbf{c}}{R}\right)$$

$$\bar{\sigma}_{sm} \equiv (1/3)\mathrm{tr}\bar{\boldsymbol{\sigma}}_s,\quad \bar{\sigma}_s^{eq} \equiv \sqrt{3/2}\,\|\bar{\boldsymbol{\sigma}}_s'^{2}\| \quad (20)$$

where R_s evolves by the following equation (see **Fig. 3**).

$$\dot{R}_s = U_s \|\dot{\boldsymbol{\varepsilon}}^{vp}\| \quad \text{for } \dot{\boldsymbol{\varepsilon}}^{vp} \neq \mathbf{O} \quad (21)$$

with

$$U_s(\mathfrak{R}_c, C_n, R_s) = u \exp(u_c \mathfrak{R}_c C_n) \cot\left(\frac{\pi}{2}\frac{\langle R_s - R_e \rangle}{1 - R_e}\right) \quad (22)$$

R_s (≤ 1) is called the static yield ratio because the quasi-static deformation proceeds in the state $R = R_s$.

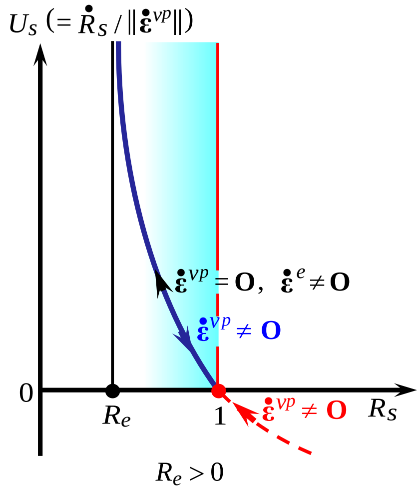

Fig. 3. Function $U_s(R_s)$ in the evolution rule of static-yield ratio R_s.

The variable $\mathfrak{R}_c$ in Eq. (22) designates the *elastic-core yield ratio*, i.e. the ratio of the size of the following *elastic-core surface*, which passes through the elastic-core $\mathbf{c}$ and is similar to the yield surface (see Fig. 2) to the size of the yield surface, while it is calculated numerically from the nonlinear equation.

$$f_c(\mathbf{c},\ \mathfrak{R}_c F,\ \xi)$$

$$= \left(\frac{c^{eq}}{\mathfrak{R}_c F}\right)^2 + 2\xi \cosh\left(\frac{3}{2}\frac{c_m}{\mathfrak{R}_c F}\right) - \xi^2 - 1 = 0 \quad (23)$$

where

$$c_m \equiv (1/3)\mathrm{tr}\mathbf{c},\quad c^{eq} \equiv \sqrt{3/2}\,\|\mathbf{c}'\| \quad (24)$$

Further, the normalized outward-normal $\hat{\mathbf{n}}_c$ of the elastic-core-surface is given from Eq. (23) as follows:

$$\hat{\mathbf{n}}_c \equiv \frac{\partial f_c}{\partial \mathbf{c}} \Big/ \left\| \frac{\partial f_c}{\partial \mathbf{c}} \right\| \quad (\|\hat{\mathbf{n}}_c\| = 1) \quad (25)$$

where

$$\frac{\partial f_c}{\partial \mathbf{c}} = \frac{2c^{eq}}{(\mathfrak{R}_c F)^2}\frac{\partial c^{eq}}{\partial \mathbf{c}} -2\xi \sinh\Big(\frac{3}{2}\frac{c_m}{\mathfrak{R}_c F}\Big)\frac{3}{2}\frac{1}{\mathfrak{R}_c F}\frac{\partial c_m}{\partial \mathbf{c}}$$
$$= \frac{2c^{eq}}{(\mathfrak{R}_c F)^2}\sqrt{\frac{3}{2}}\|\mathbf{c}'\| -2\xi \sinh\Big(\frac{3}{2}\frac{c_m}{\mathfrak{R}_c F}\Big)\frac{3}{2}\frac{1}{\mathfrak{R}_c F}\frac{1}{3}\mathbf{I}$$
$$= \frac{1}{\mathfrak{R}_c F}\Big[\sqrt{6}\frac{c^{eq}}{\mathfrak{R}_c F}\|\mathbf{c}'\| - \xi \sinh\Big(\frac{3}{2}\frac{c_m}{\mathfrak{R}_c F}\Big)\mathbf{I}\Big] \tag{26}$$

Assume the associated flow rule for the viscoplastic strain rate, referring to Hashiguchi [6] and Hashiguchi et al. [7][8]:

$$\overset{\bullet}{\boldsymbol{\varepsilon}}^{vp} = \bar{\varGamma}\bar{\mathbf{n}} \quad (\bar{\varGamma} > 0) \tag{27}$$

where the positive viscoplastic multiplier $\bar{\varGamma}$ is given by

$$\bar{\varGamma} \equiv \frac{1}{\mu_v \exp(\bar{u}_c \mathfrak{R}_c C_n)}\frac{\langle R - R_s\rangle^n}{1 - R/(c_m R_s)} \tag{28}$$

where μ_v, n, $\bar{u}_c$ and $c_m\ (\gg 1)$ are the material constants. Further, $\bar{\mathbf{n}}$ is given by

$$\bar{\mathbf{n}} \equiv \frac{\partial f(\bar{\boldsymbol{\sigma}}, R, F, \xi)}{\partial \boldsymbol{\sigma}} \Big/ \Big\| \frac{\partial f(\bar{\boldsymbol{\sigma}}, R, F, \xi)}{\partial \boldsymbol{\sigma}}\Big\| \quad (\|\bar{\mathbf{n}}\|=1) \tag{29}$$

$$\frac{\partial f(\boldsymbol{\sigma}, R, F, \xi)}{\partial \boldsymbol{\sigma}} = \frac{1}{RF}\Big[\sqrt{6}\frac{\sigma^{eq}}{RF}\frac{\bar{\boldsymbol{\sigma}}'}{\|\bar{\boldsymbol{\sigma}}'\|} - \xi \sinh\Big(\frac{3}{2}\frac{\bar{\sigma}_m}{RF}\Big)\mathbf{I}\Big] \tag{30}$$

noting

$$\frac{\partial f(\bar{\boldsymbol{\sigma}}, R, F, \xi)}{\partial \boldsymbol{\sigma}} = \frac{2\bar{\sigma}^{eq}}{(RF)^2}\frac{\partial \bar{\sigma}^{eq}}{\partial \boldsymbol{\sigma}} -2\xi \sinh\Big(\frac{3}{2}\frac{\bar{\sigma}_m}{RF}\Big)\frac{3}{2}\frac{1}{RF}\frac{\partial \bar{\sigma}_m}{\partial \boldsymbol{\sigma}}$$
$$= \frac{2\bar{\sigma}^{eq}}{(RF)^2}\frac{\partial \sqrt{3/2}\|\bar{\boldsymbol{\sigma}}'\|}{\partial \boldsymbol{\sigma}} -2\xi \sinh\Big(\frac{3}{2}\frac{\bar{\sigma}_m}{RF}\Big)\frac{3}{2}\frac{1}{RF}\frac{1}{3}\mathbf{I}$$
$$= \frac{1}{RF}\Big[\sqrt{6}\frac{\sigma^{eq}}{RF}\frac{\bar{\boldsymbol{\sigma}}'}{\|\bar{\boldsymbol{\sigma}}'\|} - \xi \sinh\Big(\frac{3}{2}\frac{\bar{\sigma}_m}{RF}\Big)\mathbf{I}\Big]$$

$$\begin{cases} \dfrac{\partial\|\boldsymbol{\sigma}'\|}{\partial \boldsymbol{\sigma}'} = \dfrac{\boldsymbol{\sigma}'}{\|\boldsymbol{\sigma}'\|}, \quad \dfrac{\partial\|\boldsymbol{\sigma}'\|}{\partial \boldsymbol{\sigma}} = \dfrac{\partial\|\boldsymbol{\sigma}'\|}{\partial \boldsymbol{\sigma}'}\dfrac{\partial \boldsymbol{\sigma}'}{\partial \boldsymbol{\sigma}} = \dfrac{\boldsymbol{\sigma}'}{\|\boldsymbol{\sigma}'\|} \\ \dfrac{\partial \bar{\sigma}_m}{\partial \sigma_{ij}} = \dfrac{\partial(\bar{\sigma}_{rs}\delta_{rs})/3}{\partial \sigma_{ij}} = \dfrac{1}{3}\delta_{ir}\delta_{js}\delta_{rs} = \dfrac{1}{3}\delta_{ij} \end{cases}$$

The variable C_n involved in the viscoplastic multiplier in Eq. (28) is given by

$$C_n \equiv \hat{\mathbf{n}}_c : \bar{\mathbf{n}} \quad (-1 \le C_n \le 1) \tag{31}$$

which is required to describe Masing effect [9], i.e. the increase and the decrease of the curvature of the stress-strain curve in the forward loading process and the inverse loading process, respectively.

8. STRESS RATE VS. STRAIN RATE RELATION

The strain rate vs. stress rate relation is given from Eqs. (4) and (27) into Eq. (2) as follows:

$$\begin{cases} \overset{\bullet}{\boldsymbol{\varepsilon}} = \mathbb{E}^{-1} : \overset{\bullet}{\boldsymbol{\sigma}} + \bar{\varGamma}\bar{\mathbf{n}} \\ \overset{\bullet}{\boldsymbol{\sigma}} = \mathbb{E} : \overset{\bullet}{\boldsymbol{\varepsilon}} - \bar{\varGamma}\mathbb{E} : \bar{\mathbf{n}} = \mathbb{E} : (\overset{\bullet}{\boldsymbol{\varepsilon}} - \bar{\varGamma}\bar{\mathbf{n}}) \end{cases} \tag{32}$$

which is represented in the incremental form as follows:

$$\begin{cases} d\boldsymbol{\varepsilon} = \mathbb{E}^{-1} : d\boldsymbol{\sigma} + \bar{\varGamma}\bar{\mathbf{n}}dt \\ d\boldsymbol{\sigma} = \mathbb{E} : d\boldsymbol{\varepsilon} - \bar{\varGamma}\mathbb{E} : \bar{\mathbf{n}}dt = \mathbb{E} : (d\boldsymbol{\varepsilon} - \bar{\varGamma}\bar{\mathbf{n}}dt) \end{cases} \tag{33}$$

9. EVOLUTIONS OF INTERNAL VARIABLRES

By incorporating the isotropic hardening stagnation formulated in **APPENDIX**, the isotropic hardening function is given as follows:

$$\begin{cases} F(H) = F_0\{1 + s_r[1 - \exp(-c_H H)]\}, \\ F' \equiv dF/dH = s_r c_H F_0 \exp(-c_H H) \\ \overset{\bullet}{H} = \tilde{R}^{\upsilon}\langle \tilde{\mathbf{n}} : \bar{\mathbf{n}}\rangle \sqrt{2/3}\|\overset{\bullet}{\boldsymbol{\varepsilon}}^{vp}\| \end{cases} \tag{34}$$

where F_0 is the initial value of F, and $s_r\ (\le 1)$ and c_H are the material constants. $\tilde{R}$ is the isotropic hardening stagnation ratio and $\tilde{\mathbf{n}}$ is the normalized outward-normal of the sub-isotropic tropic hardening surface (see APPENDIX).

The rate of the elastic-core $\mathbf{c}$ is given in APPENDIX in Hashiguchi [10] as follows:

$$\overset{\bullet}{\mathbf{c}} = c_e\{\overset{\bullet}{\boldsymbol{\varepsilon}}^p - [1 + \exp(-c_s\|\hat{\boldsymbol{\sigma}}\|/F)]\mathfrak{R}_c\hat{\mathbf{n}}_c\|\overset{\bullet}{\boldsymbol{\varepsilon}}^p\|\} \tag{35}$$

where c_e and c_s are the material constants. It is confirmed that the rate of the elastic-core in Eq. (35) never reaches the yield surface as known from the inequality

$$\hat{\mathbf{n}}_c : \overset{\bullet}{\mathbf{c}} = \hat{\mathbf{n}}_c : c_e\{\overset{\bullet}{\boldsymbol{\varepsilon}}^p - [1 + \exp(-c_s\|\hat{\boldsymbol{\sigma}}\|/F)]\mathfrak{R}_c\hat{\mathbf{n}}_c\|\overset{\bullet}{\boldsymbol{\varepsilon}}^p\|\}$$
$$= c_e\{\hat{\mathbf{n}}_c : \bar{\mathbf{n}} - [1 + \exp(-c_s\|\hat{\boldsymbol{\sigma}}\|/F)]\mathfrak{R}_c\hat{\mathbf{n}}_c : \hat{\mathbf{n}}_c\}\bar{\varGamma}$$
$$= c_e(\hat{\mathbf{n}}_c : \bar{\mathbf{n}} -])\bar{\varGamma} < 0 \quad \text{for } \mathfrak{R}_c = \frac{1}{1 + \exp(-c_s\|\hat{\boldsymbol{\sigma}}\|/F)} < 0 \tag{36}$$

noting

$$\frac{1}{1 + \exp(-c_s\|\hat{\boldsymbol{\sigma}}\|/F)}$$

$$= \begin{cases} 1/2 & \text{for } \|\hat{\boldsymbol{\sigma}}\|/F = 0 \\ \dfrac{1}{1+\exp(-c_S)} \ (>1/2) & \text{for } \|\hat{\boldsymbol{\sigma}}\|/F = 1 \\ 1 & \text{for } \|\hat{\boldsymbol{\sigma}}\|/F \to \infty \end{cases} \quad (37)$$

and thus the elastic-core reaches the yield surface for the extreme value of the material constant $c_S \to \infty$.

The evolution rate of the damage variable ξ is given by substituting Eqs. (27) and (32) into Eq. (17) as follows:

$$\mathring{\xi} = (1-\xi)\bar{\Gamma}\,\mathrm{tr}\bar{\mathbf{n}} + a_1\bar{\Gamma} + a_2\langle \mathrm{tr}[\mathbb{E}:(\mathring{\boldsymbol{\varepsilon}} - \bar{\Gamma}\bar{\mathbf{n}})]\rangle/3 \quad (38)$$

10. STRESS-UPDATE CALCULATION PROCEDURES

The stress-update calculation for the above-mentioned subloading-overstress damage model can be performed by the following procedure.

1) The viscoplastic strain increment $d\boldsymbol{\varepsilon}^{vp}$ is calculated by inputting the time-increment dt under the updated values of the viscoplastic multiplier $\bar{\Gamma}$ and the normalized-outward normal $\bar{\mathbf{n}}$ of the subloading surface in Eq. (27),
2) The internal variables, i.e. H and $\mathbf{c}$ are updated by Eqs. (34) and (35) by inputting $d\boldsymbol{\varepsilon}^{vp}$ and the time-increment dt.
3) The damage variable ξ is updated by inputting the strain rate $\mathring{\boldsymbol{\varepsilon}}$ and the time-increment dt into Eq. (38).
4) The stress is calculated by Eq. (33).

It is noticeable that this calculation procedure can be performed by the fact that the viscoplastic $\bar{\Gamma}$ does not involve any rate variable, so that it is calculated only by the state variables. On the other hand, the stress rate vs. strain rate relation must be formulated explicitly in the elastoplastic constitutive equation, while the formulation of the stress rate vs. strain rate relation is extremely complex, even if its derivation is not impossible.

11. COCLUDING REMARTKS

The elasto-viscoplastic damage constitutive equation for ductile metals is formulated based on Gurson's yield surface (Gurson [1]) in this article, which is based on the extended subloading-overstress model (Hashiguchi [6], Hashiguchi et al. [7][8]). It would be capable of describing the rate-dependent cyclic loading behavior of ductile metals rigorously.

APPENDIX:
ISOTROPIC HARDENING STAGNATION

The concept insisting that the isotropic hardening stagnates for a while after the stress reversal event was proposed by Chaboche et al. [11]. The concept insists that the isotropic hardening does not proceed when the plastic strain lies inside a certain region, called the *nonhardening region*, in the plastic strain space. The nonhardening region expands and translates when the plastic strain lies on the boundary of the region and the plastic strain rate is induced directing outwards the region.

The extended formulation for the isotropic hardening stagnation by the notion of the subloading surface concept with the overstress formulation (Hashiguchi [6]) will be given in the following.

Assuming that the isotropic hardening stagnates when the viscplastic strain $\boldsymbol{\varepsilon}^{vp}$ lies inside a certain region, let the following surface, called the *isotropic hardening stagnation surface*, be introduced.

$$g(\tilde{\boldsymbol{\varepsilon}}^{vp}) = \tilde{K} \quad (39)$$

where

$$\tilde{\boldsymbol{\varepsilon}}^{vp} \equiv \boldsymbol{\varepsilon}^{vp} - \boldsymbol{\rho} \quad (40)$$

$\tilde{K}$ and $\boldsymbol{\rho}$ designate the size and the center, respectively, of the isotropic hardening stagnation surface, the evolution rules of which will be formulated later. Here, the viscoplastic strain $\boldsymbol{\varepsilon}^{vp}$ is regarded as an internal variable. The same situation can be reminded in the Prager's lineal kinematic hardening rule.

Furthermore, we introduce the following surface, called the *sub-isotropic hardening stagnation surface*, which always passes through the current viscoplastic strain $\boldsymbol{\varepsilon}^{vp}$ and has a similar shape and a same orientation to the isotropic hardening stagnation surface.

$$g(\tilde{\boldsymbol{\varepsilon}}^{vp}) = \tilde{R}\tilde{K} \quad (41)$$

where $\tilde{R}\,(0 \le \tilde{R} \le 1)$ is the ratio of the size of sub-isotropic hardening stagnation surface to that of the isotropic hardening stagnation surface. It plays the role as the measure for the approaching degree of the plastic strain to the isotropic hardening stagnation surface. Then, $\tilde{R}$ is referred to as the *isotropic hardening stagnation ratio*. It is calculable from the equation $\tilde{R} = g(\tilde{\boldsymbol{\varepsilon}}^{vp})/\tilde{K}$ in terms of the known values of $\boldsymbol{\varepsilon}^{vp}$, $\boldsymbol{\rho}$ and $\tilde{K}$.

Now, let the function $g(\tilde{\boldsymbol{\varepsilon}}^{vp})$ be given explicitly by

$$g(\tilde{\boldsymbol{\varepsilon}}^{vp}) = \|\tilde{\boldsymbol{\varepsilon}}^{vp}\| \quad (42)$$

Then, the following relations hold.

$$\tilde{\mathbf{n}} \equiv \frac{\partial g(\tilde{\boldsymbol{\varepsilon}}^{vp})}{\partial \tilde{\boldsymbol{\varepsilon}}^{vp}} = \frac{\tilde{\boldsymbol{\varepsilon}}^{vp}}{\|\tilde{\boldsymbol{\varepsilon}}^{vp}\|} \quad (\|\tilde{\mathbf{n}}\|=1) \quad (43)$$

$$\tilde{\mathbf{n}}:\tilde{\boldsymbol{\varepsilon}}^{vp} = \frac{\tilde{\boldsymbol{\varepsilon}}^{vp}}{\|\tilde{\boldsymbol{\varepsilon}}^{vp}\|}:\tilde{\boldsymbol{\varepsilon}}^{vp} = \|\tilde{\boldsymbol{\varepsilon}}^{vp}\| \,(=1) = \tilde{R}\tilde{K} \quad (44)$$

The consistency condition for the sub-isotropic hardening stagnation surface in Eq. (41) is given by

$$\frac{\partial g(\tilde{\boldsymbol{\varepsilon}}^{vp})}{\partial \tilde{\boldsymbol{\varepsilon}}^{vp}}:\mathring{\boldsymbol{\varepsilon}}^{vp} - \frac{\partial g(\tilde{\boldsymbol{\varepsilon}}^{vp})}{\partial \tilde{\boldsymbol{\varepsilon}}^{vp}}:\mathring{\boldsymbol{\rho}} = \tilde{R}\mathring{\tilde{K}} + \mathring{\tilde{R}}\tilde{K} \quad (45)$$

which leads to the following equation by considering Eq. (43).

$$\tilde{\mathbf{n}}:(\mathring{\boldsymbol{\varepsilon}}^{vp} - \mathring{\boldsymbol{\rho}}) = \tilde{R}\mathring{\tilde{K}} + \tilde{R}\mathring{\tilde{R}} \quad (46)$$

Now, the following mechanical properties must be incorporated in the formulations of the evolution rules of the rates of $\tilde{K}$ and $\boldsymbol{\rho}$.

1) $\tilde{K}$ and $\boldsymbol{\rho}$ evolve when the plastic strain rate $\mathring{\boldsymbol{\varepsilon}}^{vp}$ is induced directing outwards the sub-isotropic hardening

stagnation surface, fulfilling

$$\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}>0,\ \langle\dot{\boldsymbol{\varepsilon}}^{vp}\rangle\neq 0 \tag{47}$$

2) The plastic strain $\boldsymbol{\varepsilon}^{vp}$ is assumed to exist inside the isotropic hardening stagnation surface. Therefore, the rate of the isotropic hardening stagnation ratio must be positive, zero and negative, when the plastic strain $\boldsymbol{\varepsilon}^{vp}$ exists inside, on and outside the isotropic hardening stagnation surface, respectively. Therefore, it must hold that

$$\left.\begin{array}{l}\dot{\tilde{R}}>0 \text{ for } \tilde{R}<1\\ \dot{\tilde{R}}=0 \text{ for } \tilde{R}=1\\ \dot{\tilde{R}}<0 \text{ for } \tilde{R}>1\end{array}\right\} \text{ for } \tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}>0, \text{ i.e. } \langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle\neq 0 \tag{48}$$

3) The rates of $\tilde{K}$ and $\boldsymbol{\rho}$ increase as the plastic strain approaches the isotropic hardening stagnation surface, i.e. as the isotropic hardening stagnation ratio $\tilde{R}$ increases. Therefore, they are monotonic-increasing function of the isotropic hardening stagnation ratio $\tilde{R}$.

4) The following relation based on the consistency condition in Eq. (46) must hold for $\dot{\tilde{R}}=0$.

$$\tilde{\mathbf{n}}:(\dot{\boldsymbol{\varepsilon}}^{vp}-\dot{\boldsymbol{\rho}})-\tilde{R}\dot{\tilde{K}}=0 \quad \text{for } \dot{\tilde{R}}=0 \tag{49}$$

Let the following evolution equations of $\tilde{K}$ and $\boldsymbol{\rho}$ which fulfill all the above-mentioned requirements.

$$\dot{\tilde{K}}=C\tilde{R}^{\varsigma-1}\langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle\ , \tag{50}$$

$$\dot{\boldsymbol{\rho}}=(1-C)\tilde{R}^{\varsigma}\langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle\tilde{\mathbf{n}}\ , \tag{51}$$

where $C\ (\leq 1)$ and $\varsigma\ (\geq 1)$ are the material constants.

The rate of the isotropic hardening stagnation ratio $\tilde{R}$ is given by substituting the evolution equations of $\tilde{K}$ and $\boldsymbol{\rho}$ into the constancy condition in Eq. (46) as follows:

$$\begin{aligned}\frac{\dot{\tilde{R}}}{\tilde{R}}&=\frac{\tilde{\mathbf{n}}:(\dot{\boldsymbol{\varepsilon}}^{vp}-\dot{\boldsymbol{\rho}})}{\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}}-\frac{\dot{\tilde{K}}}{\tilde{K}}\\ &=\frac{\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}-(1-C)\tilde{R}^{\varsigma}\langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle\tilde{\mathbf{n}}:\tilde{\mathbf{n}})}{\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}}-\frac{C\tilde{R}^{\varsigma-1}\langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle}{\tilde{K}}\\ &=\frac{1}{\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}}[\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}-(1-C)\tilde{R}^{\varsigma}\langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle)-C\tilde{R}^{\varsigma}\langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle]\\ &=\frac{1}{\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}}\langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle(1-\tilde{R}^{\varsigma}) \text{ for } \tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}>0\end{aligned} \tag{52}$$

which is the monotonically-decreasing function of $\tilde{R}$ fulfilling

$$\frac{\dot{\tilde{R}}}{\tilde{R}}\Big/\frac{\langle\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}\rangle}{\tilde{\mathbf{n}}:\dot{\boldsymbol{\varepsilon}}^{vp}}=1-\tilde{R}^{\varsigma}\left\{\begin{array}{l}=1\ (>0) \quad \text{for } \tilde{R}=0\\ <1\ (>0) \text{ for } \tilde{R}<1\\ =0 \quad \text{for } \tilde{R}=1\\ <0 \quad \text{for } \tilde{R}>1\end{array}\right. \tag{53}$$

Therefore, the isotropic hardening stagnation surface is automatically controlled to enclose the viscoplastic strain, so that large strain increments can be input in the numerical calculation.

The rate of the isotropic hardening variable based on the equivalent viscoplastic strain, i.e. $\dot{H}=\sqrt{2/3}\|\dot{\boldsymbol{\varepsilon}}^{vp}\|$ is required to be modified as follows:

$$\dot{H}=\tilde{R}^{\upsilon}\langle\tilde{\mathbf{n}}:\tilde{\mathbf{n}}\rangle\sqrt{2/3}\|\dot{\boldsymbol{\varepsilon}}^{vp}\| \tag{54}$$

where $\upsilon\ (\geq 1)$ is the material constant.

The three material constants C, ς and υ involved in the formulation of the isotropic hardening stagnation formulated above can be chosen as $C=0.5$, $\varsigma=3$ and $\upsilon=3$ actually.

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