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Unilateral Damage Model based on Subloading-overstress Model

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The unilateral damage model described in terms of the principal stress directions has been formulated by Lemaitre [1], Lemaitre and Desmoral [2], de Souza Neto et al. [3], etc. However, the calculation procedure by this unilateral damage model has been limited to the implicit stress-integration by the return-mapping (de Souza Neto et al. [3]). The subloading-viscoplastic constitutive equation with the unilateral damage will be formulated by introducing the subloading-overstress model (Hashiguchi [4], Hashiguchi et al. [5][6]) and, based on it, the explicit stress integration method will be given for the unilateral damage model in this article.

Key Words : *Forward-Euler calculation, Glass, Subloading-overstress model, Unilateral damage.*

1. INTRODUCTION

The unilateral damage in which the damage in the compressed plane is smaller than that in the tensioned plane is induced in brittle materials, e.g. cast iron, superalloy, glass, ceramics, rocks, concrete, etc. The rigorous unilateral damage model described in terms of the principal stress directions has been formulated by Lemaitre [1], Lemaitre and Desmoral [2], de Souza Neto et al. [3]. However, the calculation procedure by this unilateral model has been limited to the implicit stress-integration by the return-mapping as shown in de Souza Neto et al. [3] because the formulation of the stress rate vs. strain rate relation would be quite complex. The overstress model taken account of the unilateral damage will be formulated by incorporating the subloading-overstress model (Hashiguchi [4] Hashiguchi et al. [5][6] and, adopting it, the explicit stress-integration will be formulated in this article.

2. HYPOTHESIS OF STRAIN EQUIVALENCE FOR BRIITLE DAMAGE AND HYPERELASTIC EQUATION

The *hypothesis of strain equivalence* by Lemaitre [1] insists "*The deformation of the damaged actual material subjected to the actual stress is identical to the deformation of the undamaged material subjected to the effective stress*" is adopted. Let this hypothesis be applied also to the elastic strain $\boldsymbol{\varepsilon}^e$ and the plastic strain $\boldsymbol{\varepsilon}^{vp}$, i.e.

$$\boldsymbol{\varepsilon} = \underset{\sim}{\boldsymbol{\varepsilon}},\ \boldsymbol{\varepsilon}^e = \underset{\sim}{\boldsymbol{\varepsilon}}^e,\ \ \boldsymbol{\varepsilon}^{vp} = \underset{\sim}{\boldsymbol{\varepsilon}}^{vp} \tag{1}$$

$$\boldsymbol{\varepsilon}^e = \underset{\sim}{\mathbb{E}}^{-1} : \underset{\sim}{\boldsymbol{\sigma}} = \mathbb{E}(D)^{-1} : \boldsymbol{\sigma}(\underset{\sim}{\boldsymbol{\sigma}}, D) \quad (\|\boldsymbol{\sigma}\| \le \|\underset{\sim}{\boldsymbol{\sigma}}\|) \tag{2}$$

where the variables in the undamaged effective configuration are specified by adding the under tilde $\underset{\sim}{(\ )}$. The variable $D\ (0 \le D \le 1)$ is the damage variable. $\underset{\sim}{\boldsymbol{\sigma}}$ is called the *effective Cauchy stress* which is the average stress acting only on the undamaged area, while $\boldsymbol{\sigma}$ is the *actual Cauchy stress* which is the average stress acting on the whole area including both of the damaged and the undamaged areas.

The hyperelastic constitutive equation is described by the elastic strain energy function $\psi(\boldsymbol{\varepsilon}^e)$ and the Gibbs' free energy function $\phi(\underset{\sim}{\boldsymbol{\sigma}})$ as follows:

$$\underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial \psi(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e}, \quad \boldsymbol{\varepsilon}^e = \frac{\partial \phi(\underset{\sim}{\boldsymbol{\sigma}})}{\partial \underset{\sim}{\boldsymbol{\sigma}}} \tag{3}$$

Now, adopt the quadratic forms of $\psi(\boldsymbol{\varepsilon}^e)$ and $\phi(\underset{\sim}{\boldsymbol{\sigma}})$ as follows:

$$\left.\begin{aligned} \psi(\boldsymbol{\varepsilon}^e) &= \frac{1}{2}\boldsymbol{\varepsilon}^e : \underset{\sim}{\mathbb{E}} : \boldsymbol{\varepsilon}^e \left(= \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e\right), \\ \phi(\underset{\sim}{\boldsymbol{\sigma}}) &= \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbb{E}}^{-1} : \underset{\sim}{\boldsymbol{\sigma}} \left(= \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e\right) \end{aligned}\right\} \tag{4}$$

leading to

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\sim}{\mathbb{E}} : \boldsymbol{\varepsilon}^e, \quad \boldsymbol{\varepsilon}^e = \underset{\sim}{\mathbb{E}}^{-1} : \underset{\sim}{\boldsymbol{\sigma}} \tag{5}$$

where $\underset{\sim}{\mathbb{E}}$ is the fourth-order elastic modulus tensor in the undamaged state.

Here, let $\underset{\sim}{\mathbb{E}}$ be given by Hooke's law:

$$\left.\begin{aligned} \underset{\sim}{\mathbb{E}} &= \frac{\underset{\sim}{E}}{1+\nu}\left(\mathbb{S} + \frac{\nu}{1-2\nu}\mathbb{T}\right), \\ \underset{\sim}{\mathbb{E}}_{ijkl} &= \frac{\underset{\sim}{E}}{1+\nu}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \\ \underset{\sim}{\mathbb{E}}^{-1} &= \frac{1}{\underset{\sim}{E}}\left[(1+\nu)\mathbb{S} - \nu\mathbb{T}\right], \\ \underset{\sim}{\mathbb{E}}^{-1}_{ijkl} &= \frac{1}{\underset{\sim}{E}}\left[\frac{1}{2}(1+\nu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \nu\delta_{ij}\delta_{kl}\right] \end{aligned}\right\} \tag{6}$$

Then, the elastic strain energy function $\psi(\boldsymbol{\varepsilon}^e)$ and the elastic complementary strain energy function $\phi(\underset{\sim}{\boldsymbol{\sigma}})$ in the undamaged state are given for Eq. (6) as follows:

$$\left.\begin{aligned}\psi(\boldsymbol{\varepsilon}^e)&=\frac{1}{2}\frac{\underset{\sim}{E}}{1+\nu}\left[\varepsilon_{ij}^e\varepsilon_{ij}^e+\frac{\nu}{1-2\nu}(\varepsilon_{kk}^e)^2\right],\\ \phi(\underset{\sim}{\boldsymbol{\sigma}})&=\frac{1}{2\underset{\sim}{E}}[(1+\nu)\underset{\sim}{\sigma}_{ij}\underset{\sim}{\sigma}_{ij}-\nu(\underset{\sim}{\sigma}_{kk})^2]\end{aligned}\right\} \quad (7)$$

The effective stress $\underset{\sim}{\boldsymbol{\sigma}}$ and the elastic strain $\boldsymbol{\varepsilon}^e$ are derived from Eq. (3) with (7) as follows:

$$\left.\begin{aligned}\underset{\sim}{\boldsymbol{\sigma}}&=\frac{\partial\psi(\boldsymbol{\varepsilon}^e)}{\partial\boldsymbol{\varepsilon}^e}=\underset{\sim}{\mathbb{E}}:\boldsymbol{\varepsilon}^e=\frac{\underset{\sim}{E}}{1+\nu}\left[\boldsymbol{\varepsilon}^e+\frac{\nu}{1-2\nu}(\mathrm{tr}\,\boldsymbol{\varepsilon}^e)\mathbf{I}\right],\\ \boldsymbol{\varepsilon}^e&=\frac{\partial\phi(\underset{\sim}{\boldsymbol{\sigma}})}{\partial\underset{\sim}{\boldsymbol{\sigma}}}=\underset{\sim}{\mathbb{E}}^{-1}:\underset{\sim}{\boldsymbol{\sigma}}=\frac{1}{\underset{\sim}{E}}[(1+\nu)\underset{\sim}{\boldsymbol{\sigma}}-\nu(\mathrm{tr}\,\underset{\sim}{\boldsymbol{\sigma}})\mathbf{I}]\end{aligned}\right\} \quad (8)$$

3. HYPERELASTIC EQUATION FOR UNILATERAL DAMAGE

The cracks on the planes to which the negative (compressive) principal stresses act would close partly. This phenomenon is called the *unilateral damage*. The unilateral model in terms of the principal stresses has been formulated by Lemaitre [1], Lemaitre and Desmoral [2], de Neto Sauza et al. [3], etc.

Let the Young's modulus for the unilateral damage be given as follows (see **Fig.1**):

$$E=\begin{cases}(1-D)\underset{\sim}{E} & \text{for } \sigma\ge 0\\ (1-hD)\underset{\sim}{E} & \text{for } \sigma<0\end{cases} \quad (9)$$

where $h\,(0\le h\le 1)$ is the material constant, called the *crack closure parameter* or *partial microcrack closure*, which is $h\cong 0.2$ in many cases [1], while Poisson's ratio can be assumed to be constant.

The stress-strain relation in the uniaxial loading is given in the partial unilateral model as follows:

$$\left.\begin{aligned}&\sigma=(1-D)\underset{\sim}{E}\,\varepsilon^e,\quad \varepsilon^e=\frac{\sigma}{(1-D)\underset{\sim}{E}}\quad \text{for } \sigma\ge 0\\ &\sigma=(1-hD)\underset{\sim}{E}\,\varepsilon^e,\quad \varepsilon^e=\frac{\sigma}{(1-hD)\underset{\sim}{E}}\quad \text{for } \sigma<0\end{aligned}\right\} \quad (10)$$

Let the uniaxial stress be described as

$$\sigma^*=\sigma^++\sigma^- \quad (11)$$

where

$$\sigma^+=\langle\sigma\rangle,\quad \sigma^-=-\langle-\sigma\rangle \quad (12)$$

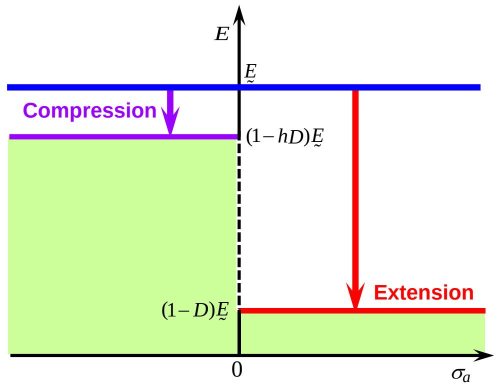
The uniaxial stress-strain relation is written as

$$\varepsilon^e=\frac{1}{\underset{\sim}{E}}\left(\frac{\sigma_+}{1-D}+\frac{\sigma_-}{1-hD}\right) \quad (13)$$

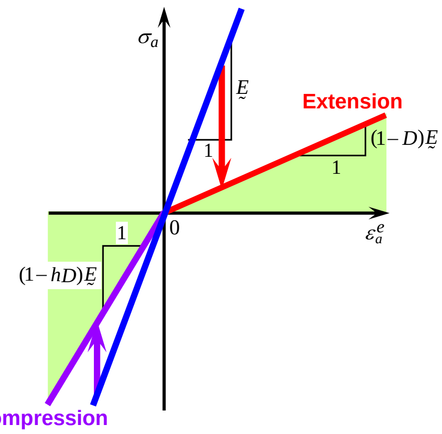
which is extended to the three-dimensional state in which the signs of principal stress are same as follows:

$$\varepsilon_i^e=\begin{cases}\dfrac{1+\nu}{\underset{\sim}{E}}\dfrac{\sigma_i}{1-D}-\dfrac{\nu}{\underset{\sim}{E}}\dfrac{\sigma_1+\sigma_2+\sigma_3}{1-D}\\ \qquad\qquad \text{for } \sigma_1,\ \sigma_2,\ \sigma_3\ge 0)\\ \dfrac{1+\nu}{\underset{\sim}{E}}\dfrac{\sigma_i}{1-hD}-\dfrac{\nu}{\underset{\sim}{E}}\dfrac{\sigma_1+\sigma_2+\sigma_3}{1-hD}\\ \qquad\qquad \text{for } \sigma_1,\ \sigma_2,\ \sigma_3<0)\end{cases} \quad (14)$$

noting Eq. (8)$_2$.



(a) Relation of principal actual Young's modulus vs. principal actual damaged stress.



(b) Relation of principal actual damage stress vs. principal elastic strain

Fig. 1 Influence of damage on Young's modulus illustrated in uniaxial loading.
(σ_a : axial stress, ε_a^e : elastic axial strain)

Now, we define the following arrays of principal stresses and elastic strains, noting that the stress is influenced by the damage but the strain is not influenced by the damage by the hypothesis of the strain equivalence described in section 2.

$$\boldsymbol{\sigma}^*\equiv\begin{bmatrix}\sigma_1^*\\ \sigma_2^*\\ \sigma_3^*\end{bmatrix},\quad \boldsymbol{\varepsilon}^{e*}\equiv\begin{bmatrix}\varepsilon_1^e\\ \varepsilon_2^e\\ \varepsilon_3^e\end{bmatrix} \quad (15)$$

where

$$\begin{bmatrix}\sigma_1^*\\ \sigma_2^*\\ \sigma_3^*\end{bmatrix}=\begin{bmatrix}\langle\sigma_1\rangle-\langle-\sigma_1\rangle\\ \langle\sigma_2\rangle-\langle-\sigma_2\rangle\\ \langle\sigma_3\rangle-\langle-\sigma_3\rangle\end{bmatrix}, \tag{16}$$

which is rewritten as

$$\boldsymbol{\sigma}^*=\boldsymbol{\sigma}_+^*+\boldsymbol{\sigma}_-^* \tag{17}$$

where $\boldsymbol{\sigma}_+^*$ and $\boldsymbol{\sigma}_-^*$ are the tensile and the compressive component, respectively, of $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma}_+^*\equiv\begin{bmatrix}\langle\sigma_1\rangle\\ \langle\sigma_2\rangle\\ \langle\sigma_3\rangle\end{bmatrix},\quad \boldsymbol{\sigma}_-^*\equiv-\begin{bmatrix}\langle-\sigma_1\rangle\\ \langle-\sigma_2\rangle\\ \langle-\sigma_3\rangle\end{bmatrix} \tag{18}$$

Further, incorporate the operators

$$\mathbf{P}_+^D=\frac{1}{1-D}\begin{bmatrix}H(\sigma_1)&0&0\\0&H(\sigma_2)&0\\0&0&H(\sigma_3)\end{bmatrix},\quad \mathbf{P}_-^D=\frac{1}{1-hD}\begin{bmatrix}\bar{H}(\sigma_1)&0&0\\0&\bar{H}(\sigma_2)&0\\0&0&\bar{H}(\sigma_3)\end{bmatrix} \tag{19}$$

where $H(s)$ is the Heaviside step function and $\bar{H}(s)$ is its opposite function, i.e.

$$H(s)=\begin{cases}1 & \text{for } s>0\\ 0 & \text{for } s\le 0\end{cases},\quad \bar{H}(s)=\begin{cases}0 & \text{for } s>0\\ 1 & \text{for } s\le 0\end{cases} \tag{20}$$

In addition, incorporate the identity tensor for the unilateral damage:

$$\mathbf{I}_+^D=\frac{H(\mathrm{tr}\boldsymbol{\sigma})}{1-D}\mathbf{I},\quad \mathbf{I}_-^D=\frac{\bar{H}(\mathrm{tr}\boldsymbol{\sigma})}{1-hD}\mathbf{I} \tag{21}$$

where

$$\mathbf{I}=\begin{bmatrix}1\\1\\1\end{bmatrix} \tag{22}$$

while the following expression holds.

$$\mathrm{tr}\boldsymbol{\sigma}=\mathbf{I}^T\boldsymbol{\sigma}^* \tag{23}$$

The elastic constitutive equation is given by modifying Eq. (5) to the unilateral damage:

$$\boldsymbol{\varepsilon}^{e*}=\mathbb{E}^{*-1}\boldsymbol{\sigma}^*,\quad \boldsymbol{\sigma}^*=\mathbb{E}^*\boldsymbol{\varepsilon}^{e*} \tag{24}$$

where

$$\mathbb{E}^*=\Big[\frac{1+\nu}{\underset{\sim}{E}}(\mathbf{P}_+^D+\mathbf{P}_-^D)-\frac{\nu}{\underset{\sim}{E}}(\mathbf{I}_+^D+\mathbf{I}_-^D)\mathbf{I}^T\Big]^{-1} \tag{25}$$

leading to

$$\boldsymbol{\varepsilon}^{e*}=\Big[\frac{1+\nu}{\underset{\sim}{E}}(\mathbf{P}_+^D+\mathbf{P}_-^D)-\frac{\nu}{\underset{\sim}{E}}(\mathbf{I}_+^D+\mathbf{I}_-^D)\mathbf{I}^T\Big]\boldsymbol{\sigma}^*,\quad \boldsymbol{\sigma}^*=\Big[\frac{1+\nu}{\underset{\sim}{E}}(\mathbf{P}_+^D+\mathbf{P}_-^D)-\frac{\nu}{\underset{\sim}{E}}(\mathbf{I}_+^D+\mathbf{I}_-^D)\mathbf{I}^T\Big]^{-1}\boldsymbol{\varepsilon}^{e*} \tag{26}$$

while the upper equation is described in the index notations as follows:

$$\begin{bmatrix}\varepsilon_1^e\\ \varepsilon_2^e\\ \varepsilon_3^e\end{bmatrix}=\left(\frac{1+\nu}{\underset{\sim}{E}}\left\{\frac{1}{1-D}\begin{bmatrix}H(\sigma_1)&0&0\\0&H(\sigma_2)&0\\0&0&H(\sigma_3)\end{bmatrix}+\frac{1}{1-hD}\begin{bmatrix}\bar{H}(\sigma_1)&0&0\\0&\bar{H}(\sigma_2)&0\\0&0&\bar{H}(\sigma_3)\end{bmatrix}\right\}-\frac{\nu}{\underset{\sim}{E}}\left\{\frac{H(\sigma_1+\sigma_2+\sigma_3)}{1-D}\begin{bmatrix}1\\1\\1\end{bmatrix}+\frac{\bar{H}(\sigma_1+\sigma_2+\sigma_3)}{1-hD}\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}[1\ 1\ 1]\right)\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix} \tag{27}$$

4. ELASTO-VISCOPLASTIC DAMAGE MODEL WIH SUBLOADING SURFACE MODEL

It should be noticed that the time-derivative of Eq. (24) for the unilateral elastic constitutive equation would be quite complex. Then, the formulation for the elastoplastic constitutive equation by use of the unilateral elastic constitutive equation in the principal stress direction is quite difficult, and thus the implicit calculation method by the return-mapping was proposed by de Souza Neto et al. [3]. However, the explicit stress integration (forward-Euler method) can be performed simply by adopting the initial subloading-overstress model. Here, it is noticeable that the rate variable is not involved in the equation of the viscoplastic strain rate and that the elastoplastic deformation can be described accurately as the quasi-static deformation by the overstress model (Hashiguchi [4], Hashiguchi et al. [5][6]). Then, the explicit stress-integration will be shown for the analysis of the unilateral damage phenomenon by adopting the initial subloading-viscoplastic model with the isotropic hardening but without the translations of the kinematic hardening variable and the elastic-core, noting that the viscoplastic deformation is rather small in the brittle damage.

Let the yield surface be given by

$$f(\boldsymbol{\sigma})=F(\varepsilon_v^{vp},\ \varepsilon^{vp\prime}) \tag{28}$$

and the subloading surface which is similar to the yield surface with respect to the origin of the stress space, i.e. the null stress and passes through the current stress, is described as

$$f(\boldsymbol{\sigma})=RF(\varepsilon_v^{vp},\ \varepsilon^{vp\prime}),\quad R=f(\boldsymbol{\sigma})/F(\varepsilon_v^{vp},\ \varepsilon^{vp\prime}) \tag{29}$$

where R is the ratio of the size of the subloading surface to that of the yield surface and

$$\left.\begin{aligned}&\varepsilon_v^{vp} \equiv \mathrm{tr}\boldsymbol{\varepsilon}^{vp}, \quad \boldsymbol{\varepsilon}^{vp\prime} \equiv \boldsymbol{\varepsilon}^{vp} - (\varepsilon_v^{vp}/3)\mathbf{I}, \\ &\varepsilon^{vp\prime} \equiv \sqrt{2/3}\,\|\boldsymbol{\varepsilon}^{vp\prime}\|\end{aligned}\right\} \tag{30}$$

The viscoplastic strain rate is given by

$$\boxed{\dot{\boldsymbol{\varepsilon}}^{vp} = \Gamma \mathbf{n}} \tag{31}$$

where

$$\mathbf{n} = \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \right\| \tag{32}$$

And the viscoplastic multiplier Γ is given by

$$\Gamma \equiv \frac{1}{\mu_v}\langle R - R_s \rangle^n \tag{33}$$

where μ_v and n are the material constants. The rate of the static yield ratio R_s must satisfy the condition

$$\dot{R}_s \begin{cases} \to \infty & \text{for } R_s = 0 \\ > 0 & \text{for } 0 < R_s < 1 \ \text{ for } \dot{\boldsymbol{\varepsilon}}^{vp*} \neq \mathbf{O} \\ = 0 & \text{for } R_s = 1 \end{cases} \tag{34}$$

Then, let its evolution equation be given as

$$\begin{cases} \dot{R}_s = U_s \|\dot{\boldsymbol{\varepsilon}}^{vp*}\| = U_s \Gamma & \text{for } \dot{\boldsymbol{\varepsilon}}^{vp*} \neq \mathbf{O} \\ R_s = R & \text{for other} \end{cases} \tag{35}$$

where

$$U_s(R_s) = u \cot\left(\frac{\pi}{2} R_s\right) \tag{36}$$

where u is the material constants.

It is noticeable that Eq. (31) for the viscoplastic strain rate involves only the current state variables but does not involve any rate variable.

5. EVOLUTION OF DAMAGE VARIABLES

The damage variable D is interpreted as an indirect measure of density of microcracks [7]. Let the evolution rule of D be given by the following equation which is the extension of the equation for the elastoplastic deformation (Lemaitre [8]) to the viscoplastic deformation.

$$\boxed{\dot{D} = \left(\frac{-Y}{\zeta}\right)^a \frac{H(\varepsilon^{vp} - \varepsilon_D^{vp})}{1-D} \|\dot{\boldsymbol{\varepsilon}}^{vp}\|} \tag{37}$$

i.e.

$$\dot{D} = \left(\frac{-Y}{\zeta}\right)^a \frac{H(\varepsilon^{vp} - \varepsilon_D^{vp})}{1-D} \Gamma \tag{38}$$

where ζ and a are the material constants, and ε_D^{vp} is the threshold value of the accumulation of the plastic strain, i.e. $\varepsilon^{vp} \equiv \int \|\dot{\boldsymbol{\varepsilon}}^{vp}\| dt$. Y is the *strain energy release rate* as shown in the following.

The free energy function for the unilateral damage is given by

$$\phi^D(\boldsymbol{\sigma}, D) = \frac{1+\nu}{2\underset{\sim}{E}}\left(\frac{\boldsymbol{\sigma}_+^* : \boldsymbol{\sigma}_+^*}{1-D} + \frac{\boldsymbol{\sigma}_-^* : \boldsymbol{\sigma}_-^*}{1-hD}\right)$$

$$-\frac{\nu}{2\underset{\sim}{E}}\left(\frac{\langle \mathrm{tr}\boldsymbol{\sigma}\rangle^2}{1-D} + \frac{\langle -\mathrm{tr}\boldsymbol{\sigma}\rangle^2}{1-hD}\right) \tag{39}$$

which is obtained by extending the strain energy function in Eq. $(7)_2$ .by taking account of Eq. (9). Then, the strain energy release rate is given as

$$Y = -\frac{\partial \phi^D(\boldsymbol{\sigma}, D)}{\partial D} = -\frac{1}{2\underset{\sim}{E}(1-D)^2}[(1+\nu)\boldsymbol{\sigma}_+ : \boldsymbol{\sigma}_+ - \nu\langle \mathrm{tr}\boldsymbol{\sigma}\rangle^2]$$

$$-\frac{h}{2\underset{\sim}{E}(1-hD)^2}[(1+\nu)\boldsymbol{\sigma}_- : \boldsymbol{\sigma}_- - \nu\langle -\mathrm{tr}\boldsymbol{\sigma}\rangle^2] \tag{40}$$

by Lemaitre [1] and quoted by Lemaitre and Desmoral [2], de Souza Neto et al. [3], etc.

6. NUMERICAL STRESS INTGRATION

The numerical calculation is performed by the following procedure.

1) The elastic strain tensor $\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{vp}$ $(\varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^{vp})$ is calculated by inputting the new strain tensor $\boldsymbol{\varepsilon}\ (\varepsilon_{ij})$. Then, the principal elastic strain vector $\boldsymbol{\varepsilon}^{e*}\ (\varepsilon_{\mathrm{i}}^{e*})$ is calculated from the elastic strain tensor $\boldsymbol{\varepsilon}^e\ (\varepsilon_{ij}^e)$.

2) The principal stress vector $\boldsymbol{\sigma}^*\ (\sigma_{\mathrm{i}}^*)$ is calculated from the principal elastic strain vector $\boldsymbol{\varepsilon}^{e*}\ (\varepsilon_{\mathrm{i}}^{e*})$ by Eq. (26). Then, the stress tensor $\boldsymbol{\sigma}\ (\sigma_{ij})$ in the prescribed coordinate system is calculated from $\boldsymbol{\sigma}^*\ (\sigma_{\mathrm{i}}^*)$.

3) The viscoplastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}^{vp}\ (\dot{\varepsilon}_{ij}^{vp})$ is calculated from Eq. (31) by inputting the stress tensor $\boldsymbol{\sigma}\ (\sigma_{ij})$, the yield ratio R and the static yield ratio R_s.

The above calculation steps 1)-3) are repeated performing the following updating procedures of the involved variables.

4) The viscoplastic strain tensor is updated by $\boldsymbol{\varepsilon}^{vp} + \dot{\boldsymbol{\varepsilon}}^{vp} dt\ (\varepsilon_{ij}^{vp} dt + \dot{\varepsilon}_{ij}^{vp} dt)$.

5) The rate of the static yield ratio $\dot{R}_s$ is calculated by inputting the viscoplastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}^{vp}$ into Eq. (35), and then the static yield ratio is updated by $R_s + \dot{R}_s\, dt$.

6) The rate of the damage variable, $\dot{D}$, is calculated by inputting the viscoplastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}^{vp}$ into Eq. (37) with Eq. (40), and then the damage variable is updated by $D + \dot{D}\, dt$.

7) The inverse of elastic modulus $\mathbb{E}_{\mathrm{ij}}^{*-1}$ in Eq. (25) for the principal stress directions is updated by inputting the updated damage variable D.

8) The yield ratio R is updated by substituting the updated stress tensor $\boldsymbol{\sigma}$ and the updated viscoplastic strain tensor $\boldsymbol{\varepsilon}^{vp}$ into Eq. (29).

The calculation is proceeded by repeating the above-mentioned calculation procedure.

7. CONCLUDING REMARKS

The subloading-overstress model with the unilateral damage was formulated and, based on it, the explicit stress integration method was proposed in this article. It would be applicable to the description of the unilateral damage phenomena of various brittle materials, i.e. rock, concrete and ceramics, cast iron, etc.

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