Proceedings of Computational Engineering Conference JSCES
Vol. 28, 2023 May

Extension of Mullins effect for cyclic loading by subloading surface model

Koichi Hashiguchi
Dr. Eng., Solids & Structure Solutions, Ltd. (Emeritus Professor of Kyushu University)
(6-13-1002, Kuromon, Chuo-ku, Fukuoka, Japan, E-mail: hashikoi87@gmail.com)

Most of rubber-like materials consist of many fibers with a distribution of small carbon particles as a filler. Their strength decreases with a deformation since fibers are cut with a tensile deformation. This phenomenon is called the *Mullins effect* (Mullins [1][2]). The viscoelastic constitutive equation with the Mullins effect is formulated extending the formulation by Simo [5] and Miehe [6] by incorporating the subloading surface concept.

Key Words : *Damage, Mullins effect, Polymer, Subloading surface, Viscoelasticity*

1. INTRODUCTION

Most of rubber-like materials consist of many fibers with a distribution of small carbon particles as a filler. Their strength decreases with a deformation since the fibers are cut with a tensile deformation. This phenomenon is called the *Mullins effect* (Mullins [1][2]).That is, the Mullins effect is a particular aspect of the mechanical response in filled rubbers, in which the stress–strain curve is influenced by the maximum loading previously experienced causing the cuts of the fibers. Various constitutive equations have been proposed by Fung [3], Simo [5], Miehe [6], Ogden and Roxburgh [7], de Souza Neto et al. [8], etc. Among them, the models other than the one proposed by Simo [5] and Miehe [6] are physically unacceptable, since the irreversibility is not considered but they are merely the slight modifications of elastic constitutive equations, called the pseudo-elasticity by Fung [3]. In addition, the formulation by de Souza Neto et al. [8] is physically irrelevant since the purely-elastic deformation is assumed in the initial loading process, although in fact the fibers are cut in the initial loading process. On the other hand, the model proposed by Simo [5] and Miehe [6] possesses the basic structure for the irreversible deformation. However, it falls within the conventional plasticity-like framework (Drucker [9]) so that the damage evolves abruptly when the damage variable reaches the maximum loading experienced in the past and the accumulation of the damage during the cyclic loading lower than the past maximum loading cannot be described.

In this article, the formulation of Simo [5] and Miehe [6] is extended so as to describe the gradual development of the damage as the damage variable approaches the past maximum loading and the evolution of the damage variable during the cyclic loading process by incorporating the concept of the *subloading surface* ([10][11][12]). The essentials of the formulations was described in Hashiguchi [12].

2. VISCOPLASTIC RHEPLOGICAL MODEL OF POLYMERS

The viscoelastic rheological model for the deformation of polymers is shown in **Fig. 1**, where the *Prony series* is adopted for the viscoelastic deformation. It is composed of parallel series of the one purely-elastic part represented by the spring E_∞ and the arbitrary number m of Maxwell models. The spring $E_1,\ E_2,\ \cdots,\ E_m$ denotes the purely-elastic part induced in the equilibrium state after the elapse of infinite time $t \to \infty$. The Maxwell models describe the purely elastic deformation and the viscous deformation induced in the non-equilibrium state. Here, it is postulated that the former causes both the volumetric and the isochoric (volume-preserving) deformation but the latter causes only the isochoric deformation based on the experimental observation. The volumetric part and the isochoric part are designated by $(\ )_{vol}$ and $(\ )_{iso}$, respectively.

Viscoelastic deformation
(generalized Maxwell model)

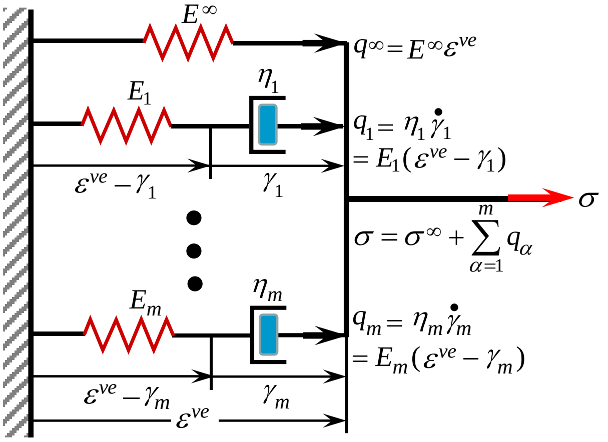

Fig. 1 Viscoelastic rheological model.

3. SUBLOADING OVERSTRESS-MULLINS EFFECT

Let the viscoelastic model with the Mullins effect [1][2] be formulated by incorporating the subloading surface concept [10][11][12] into the formulation by Simo [5] and Miehe [6].

The elastic strain energy function was extended to the viscoelastic-damage phenomenon by Holzapfel [13], assuming that the damage affects only the isochoric part of the deformation as follows:

$$\Psi^{\zeta}(\bar{\mathbf{C}}^{ve},\ \zeta, \bar{\mathbf{\Gamma}}_1, \bar{\mathbf{\Gamma}}_2,\ \cdots,\ \bar{\mathbf{\Gamma}}_m) = \Psi^{\infty}_{vol}(J^{ve})$$
$$+(1-\zeta)[\Psi^{\infty}_{iso}(\underline{\bar{\mathbf{C}}}^{ve}) + \sum_{\alpha=1}^{m}\Upsilon_{iso\alpha}(\underline{\bar{\mathbf{C}}}^{ve},\ \bar{\mathbf{\Gamma}}_{\alpha})] \tag{1}$$

where

$$\mathbf{F} = \mathbf{F}^{ve}\mathbf{F}^{vp} \tag{2}$$

$$\begin{cases} \mathbf{F}^{ve} = \mathbf{F}^{ve}_{vol}\underline{\mathbf{F}}^{ve}, \\ \mathbf{F}^{ve}_{vol} \equiv J^{ve1/3}\mathbf{g},\quad \underline{\mathbf{F}}^{ve} \equiv J^{ve-1/3}\mathbf{F}^{ve},\quad J^{ve} \equiv \det\mathbf{F}^{ve} \end{cases} \tag{3}$$

$$\mathbf{C} \equiv \mathbf{F}^T\mathbf{F} = \mathbf{F}^{vpT}\bar{\mathbf{C}}^{ve}\mathbf{F}^{vp} \tag{4}$$

$$\bar{\mathbf{C}}^{ve} \equiv \mathbf{F}^{veT}\mathbf{F}^{ve} = (\mathbf{F}\mathbf{F}^{vp-1})^T\mathbf{F}\mathbf{F}^{vp-1} = \mathbf{F}^{vp-T}\mathbf{C}\mathbf{F}^{vp-1},$$
$$\bar{\mathbf{C}}^{ve} = \bar{\mathbf{C}}^{ve\ T}_{vol}\underline{\bar{\mathbf{C}}}^{ve} \tag{5}$$

$$\begin{cases} \bar{\mathbf{C}}^{ve}_{vol} \equiv \mathbf{F}^{ve\ T}_{vol}\mathbf{F}^{ve}_{vol} = (\det\bar{\mathbf{C}}^{ve})^{1/3}\bar{\mathbf{G}} \\ \underline{\bar{\mathbf{C}}}^{ve} \equiv \underline{\mathbf{F}}^{veT}\underline{\mathbf{F}}^{ve} = (\det\bar{\mathbf{C}}^{ve})^{-1/3}\bar{\mathbf{C}}^{ve} \end{cases} \tag{6}$$

$\mathbf{F}$ is the deformation gradient tensor, and $\mathbf{F}^{ve}$ and $\mathbf{F}^{vp}$ are the viscoelastic and the viscoplastic parts of the deformation gradient tensor, respectively. $\bar{\mathbf{\Gamma}}_{\alpha}$ $(\alpha$=1, 2, $\cdots$, $m)$ are the internal variables denoting the viscoelastic deformation histories. The variables in the intermediate configuration $\bar{\mathcal{K}}$ is designated by adding the over-bar $(\bar{\ })$ and the isochoric part by the under-bar $(\underline{\ })$. $\mathbf{g}$ is the metric tensor in the current configuration. $\mathbf{C}$ is the right Cauchy-Green deformation tensor in the reference configuration. $\bar{\mathbf{C}}^{ve}$ is the viscoelastic right Cauchy-Green deformation tensor and $\bar{\mathbf{G}}$ is the metric tensor in the intermediate configuration. ζ is the damage variable the evolution rule of which was given by Miehe [6] simplifying the formulation of Simo [5] as follows (**Fig. 2**):

$$\zeta(\gamma_d) = \zeta_{\infty}[1-\exp(-\gamma_d/\upsilon)] \tag{7}$$

$$\zeta'(\gamma_d) = \frac{\partial\zeta(\gamma_d)}{\partial\gamma_d} = \zeta_{\infty}\frac{\gamma_d}{\upsilon}\exp\left(-\frac{\gamma_d}{\upsilon}\right) \tag{8}$$

where ζ_{∞} denotes the maximum value of the Mullins-damage variable ζ and υ is the material constant regulating the evolution speed of the Mullins-damage variable. The variable γ_d evolves when $\bar{\mathbf{C}}^{ve}$ lies on the Mullins-damage surface defined by

$$\Phi(\bar{\mathbf{C}}^{ve}) - \gamma_d = 0 \tag{9}$$

and the surface expands so that the evolution rule is given by

$$\dot{\gamma}_d = \dot{\Phi}(\bar{\mathbf{C}}^{ve}) = \frac{\partial\Phi(\bar{\mathbf{C}}^{ve})}{\partial\bar{\mathbf{C}}^{ve}}:\dot{\bar{\mathbf{C}}}^{ve} > 0 \tag{10}$$

The variable γ_d is induced discontinuously when it reaches the Mullins-damage surface and in addition the input-incremental step must be taken to be infinitesimal in the numerical calculation such that γ_d does not go out finitely from the Mullins-damage surface. In what follows, the revised evolution rule of the variable γ_d in which these shortcomings in the past formulations are excluded will be formulated by incorporating the subloading surface concept (Hashiguchi [10][11][12]).

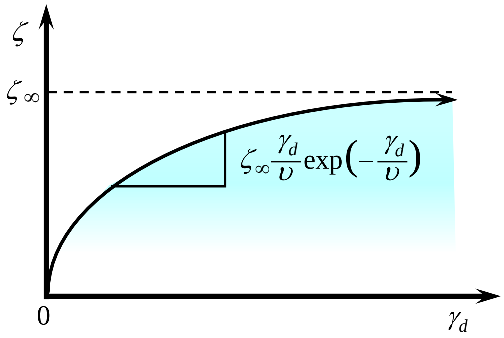

Fig. 2 Evolution of damage variable.

The surface defined in Eq. (9) is renamed as the *normal-Mullins damage surface* and the *subloading-Mullins damage surface*

$$\boxed{\Phi(\bar{\mathbf{C}}^{ve}) - R_d\gamma_d = 0 \rightarrow R_d = \Phi(\bar{\mathbf{C}}^{ve})/\gamma_d} \tag{11}$$

is incorporated (**Fig. 3**), where R_d $(0 \leq R_d \leq 1)$ is called the *normal-Mullins damage ratio* designating the approaching degree of $\bar{\mathbf{C}}^{ve}$ to the normal-Mullins damage surface in Eq. (9). The consistency condition for the subloading-Mullins surface is given by

$$\frac{\partial\Phi(\bar{\mathbf{C}}^{ve})}{\partial\bar{\mathbf{C}}^{ve}}:\dot{\bar{\mathbf{C}}}^{ve} - \dot{R}_d\gamma_d - R_d\dot{\gamma}_d = 0 \tag{12}$$

Then, assume the following evolution rule of the variable γ_d.

$$\boxed{\dot{\gamma}_d = R_d^n\langle\dot{\Phi}(\bar{\mathbf{C}}^{ve})\rangle = R_d^n\left\langle\frac{\partial\Phi(\bar{\mathbf{C}}^{ve})}{\partial\bar{\mathbf{C}}^{ve}}:\dot{\bar{\mathbf{C}}}^{ve}\right\rangle} \tag{13}$$

where n (≥ 1) is the material constant. The rate of R_d is given from Eq. (12) with Eq. (13) as follow:

$$\dot{R}_d = \frac{1}{\gamma_d}\left[\frac{\partial\Phi(\bar{\mathbf{C}}^{ve})}{\partial\bar{\mathbf{C}}^{ve}}:\dot{\bar{\mathbf{C}}}^{ve} - R_d^{n+1}\left\langle\frac{\partial\Phi(\bar{\mathbf{C}}^{ve})}{\partial\bar{\mathbf{C}}^{ve}}:\dot{\bar{\mathbf{C}}}^{ve}\right\rangle\right]$$
$$= \frac{1}{\gamma_d}\left\langle\frac{\partial\Phi(\bar{\mathbf{C}}^{ve})}{\partial\bar{\mathbf{C}}^{ve}}:\dot{\bar{\mathbf{C}}}^{ve}\right\rangle(1-R_d^{n+1})$$
$$\text{for } \frac{\partial\Phi(\bar{\mathbf{C}}^{ve})}{\partial\bar{\mathbf{C}}^{ve}}:\dot{\bar{\mathbf{C}}}^{ve} \geq 0 \tag{14}$$

which is the monotonically-increasing function of R_d fulfilling

$$\dot{R}_d \begin{cases} = \dfrac{1}{\gamma_d}\left\langle \dfrac{\partial \Phi(\bar{\mathbf{C}}^{ve})}{\partial \bar{\mathbf{C}}^{ve}} : \dot{\bar{\mathbf{C}}}^{ve} \right\rangle \text{ for } R_d = 0 \\ < \dfrac{1}{\gamma_d}\left\langle \dfrac{\partial \Phi(\bar{\mathbf{C}}^{ve})}{\partial \bar{\mathbf{C}}^{ve}} : \dot{\bar{\mathbf{C}}}^{ve} \right\rangle (>0) \text{ for } R_d < 1 \\ = 0 \quad \text{for } R_d = 1 \\ < 0 \quad \text{for } R_d > 1 \end{cases} \tag{15}$$

Therefore, the normal-Mullins damage surface evolves automatically so as to envelope variable $\bar{\mathbf{C}}^{ve}$ always. The smooth transition from the non-evolution state to the normal-evolution state of damage can be described by incorporation of the above-mentioned formulation based on the subloading surface concept as shown in **Fig. 4**.

Besides, the Mullins-damage function $\Phi(\bar{\mathbf{C}}^{ve})$ was given by Holzapfel [13]) as follows:

$$\begin{aligned}\Phi(\bar{\mathbf{C}}^{ve}) &= C_1[\mathrm{I}_c(\bar{\mathbf{C}}^{ve})-3]+C_2[\mathrm{II}_c(\bar{\mathbf{C}}^{ve})-3] \\ &= C_1(\mathrm{tr}\bar{\mathbf{C}}^{ve}-3)+C_2[(1/2)(\mathrm{tr}^2\bar{\mathbf{C}}^{ve}-\mathrm{tr}\bar{\mathbf{C}}^{ve2})-3]\end{aligned} \tag{16}$$

in the Mooney-Rivlin type [14], where C_1 and C_2 are the material constants.

The stress $\bar{\mathbf{S}}$ for Eq. (1) is given as follows:

$$\begin{aligned}\bar{\mathbf{S}} &= 2\frac{\partial \Psi^{\zeta}(\bar{\mathbf{C}}^{ve},\ \zeta, \bar{\mathbf{\Gamma}}_1, \bar{\mathbf{\Gamma}}_2, \cdots, \bar{\mathbf{\Gamma}}_m)}{\partial \bar{\mathbf{C}}^{ve}} \\ &= \bar{\mathbf{S}}^{\infty}_{vol}(J^{ve})+(1-\zeta)[\bar{\mathbf{S}}^{\infty}_{iso}(\bar{\mathbf{C}}^{ve})+\sum_{\alpha=1}^{m}\bar{\mathbf{Q}}_{\alpha}]\end{aligned} \tag{17}$$

which is rewritten as follows:

$$\begin{aligned}\bar{\mathbf{S}} = {} & J^{ev}\frac{d\Psi^{\infty}_{vol}(J^{ev})}{dJ^{ev}}\bar{\mathbf{C}}^{ev-1} \\ & +(1-\zeta)\Big\{J^{ve-2/3}\,\mathbb{P}^{ve}:\Big[2\frac{\partial \Psi^{\infty}_{iso}(\bar{\mathbf{C}}^{ve})}{\partial \bar{\mathbf{C}}^{ve}}\Big] \\ & +\sum_{\alpha=1}^{m}\Big[\exp\Big(-\frac{T}{\tau_\alpha}\Big)\bar{\mathbf{Q}}_{\alpha 0}+\int_0^T \exp\Big(-\frac{T-t}{\tau_\alpha}\Big)\beta_\alpha \dot{\bar{\mathbf{S}}}^{\infty}_{iso}(t)dt\Big]\Big\}\end{aligned} \tag{18}$$

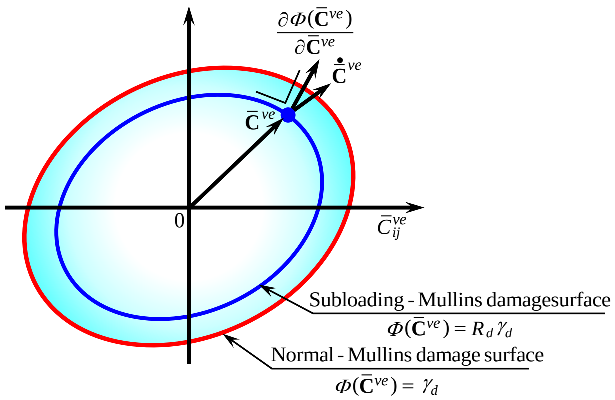

Fig. 3 Normal- and subloading-Mullins damage surfaces.

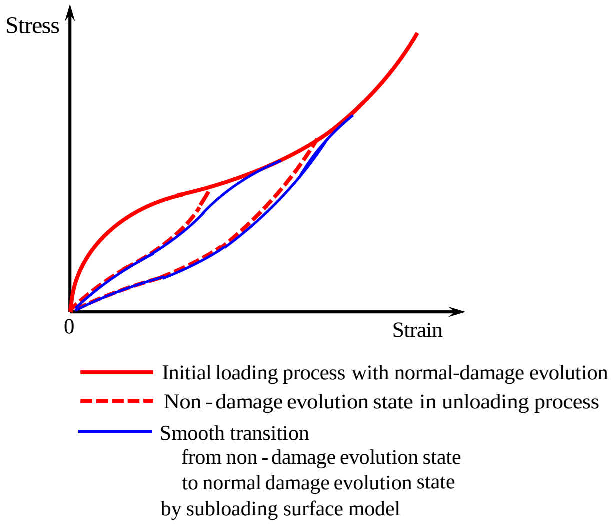

Fig. 4 Stress-strain curve with subloading-Mullins effect.

ACKNOWLEDGMENT: The author expresses the gratitude to Prof. Junji Yoshida, Yamanashi Univ. for reviewing the past researches on the deformation of the polymers and the Mullins effect.

REFERENCES

[1] Mullins, L. (1947): Effect of stretching on the properties of rubber, *J. Rubber Research*, **16**, 275-289.

[2] Mullins, L. (1947): Effect of stretching on the properties of rubber, *J. Rubber Research*, **16**, 275-289.

[3] Fung, Y. C. (1980): On the pseudo-elasticity of living tissues, *In Mechanics Today* (ed. S. N. Nasser), Ch. 4, pp.49-66, Pergamon.

[4] Gurtin, M. E. and Francis, E. C. (1981): Simple rate-independent model for damage, *J. Spacecraft*, **18**, 285-286.

[5] Simo, J.C. (1987): On a fully three-dimensional finite-stain viscoelastic damage model: formulation and computational aspects, *Comput. Meth. Appl. Mech. Eng.*, **60**, 153-173.

[6] Miehe, C. (1995): Discontinuous and continuous damage evolution in Ogden-type large-strain elastic materials, *European J. Mech., A/*Solids, **14**, 697-720.

[7] Ogden, W. and Roxburgh, D. G. (1999): Pseudo-elastic model for the Mullins effect in filled rubber, Proc. Royal Society London, **A455**, 2861-2877.

[8] de Souza Neto, E. A., Peric, D. and Owen, D. R. J.: (1994): A phenomenological three dimensional rate-independent continuum damage model for highly-filled Polymers: Formulation and Computational aspects, *J. Mech. Phys. Solids*, **42**, 1533-1550.

[9] Drucker, D. C. (1988): Conventional and unconventional plastic response and representation, *Appl. Mech. Rev.* (ASME), **41**, 151-167.

[10] Hashiguchi, K. (1980): Constitutive equations of

elastoplastic materials with elastic-plastic transition, *J. Appl. Mech*. (ASME), **47**, 266-272.

[11] Hashiguchi, K. (1989): Subloading surface model in unconventional plasticity, *Int. J. Solids Structures*, **25**, 917-945.

[12] Hashiguchi, K. (2023): *Foundations of Elastoplasticity: Subloading Surface Model*, Springer.

[13] Holzapfel, G. A. (2000): *Nonlinear Solid Mechanics: A Continuum Approach for Engineering*, John Wiley & Sons, Ltd.

[14] Mooney, M. (1940): A theory of large elastic deformation, *J. Appl. Phys*., **11**(9), pp. 582-592.