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カルマンフィルタによる散乱逆問題の再構成について

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We study the inverse medium scattering problem to reconstruct the unknown inhomogeneous medium from the far-field patterns of scattered waves. The inverse scattering problem is generally ill-posed and nonlinear, and the iterative optimization method is often adapted. A natural iterative approach to this problem is to place all available measurements and mappings into one long vector and mapping, respectively, and to iteratively solve the linearized large system equation using the Tikhonov regularization method, which is called the Levenberg-Marquardt scheme. However, this is computationally expensive because we must construct the larger system equations when the number of available measurements increases. In this paper, we propose two reconstruction algorithms based on the Kalman filter. One is the algorithm equivalent to the Levenberg-Marquardt scheme, and the other is inspired by the Extended Kalman Filter. For the algorithm derivation, we iteratively apply the Kalman filter to the linearized equation for our nonlinear equation. Our proposed algorithms sequentially update the state and the weight of the norm for the state space, which avoids the construction of a large system equation and retains the information of past updates. Finally, we provide numerical examples to demonstrate our proposed algorithms.

Key Words : *Inverse acoustic scattering, Inhomogeneous medium, Far-field pattern, Tikhonov regularization method, Levenberg–Marquardt, Kalman filter, Extended Kalman filter*

1. Introduction

Let $k > 0$ be the wave number, and let $\theta \in \mathbb{S}^1$ be incident direction. We denote the incident field $u^{inc}(\cdot, \theta)$ with the direction θ by the plane wave of the form

$$u^{inc}(x, \theta) := \mathrm{e}^{ikx\cdot\theta},\ x \in \mathbb{R}^2. \tag{1}$$

Let $Q \subset \mathbb{R}^2$ be a bounded open set with the smooth boundary and let its exterior $\mathbb{R}^2 \setminus \overline{Q}$ be connected. We assume that $q \in L^\infty(\mathbb{R}^2)$, which refers to the inhomogeneous medium, satisfies $\mathrm{Re}(1 + q) > 0$, $\mathrm{Im}q \geq 0$, and its support supp q is embed into Q, that is supp $q \Subset Q$. Then, the direct scattering problem is to determine the total field $u = u^{sca} + u^{inc}$ such that

$$\Delta u + k^2(1 + q)u = 0 \text{ in } \mathbb{R}^2, \tag{2}$$

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u^{sca}}{\partial r} - iku^{sca}\Big) = 0, \tag{3}$$

where $r = |x|$. The *Sommerfeld radiation condition* (3) holds uniformly in all directions $\hat{x} := \frac{x}{|x|}$. Furthermore, the problem (2)–(3) is equivalent to the *Lippmann-Schwinger integral equation*

$$u(x, \theta) = u^{inc}(x, \theta) + k^2 \int_Q q(y)u(y, \theta)\Phi(x, y)dy, \tag{4}$$

where $\Phi(x, y)$ denotes the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, that is,

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|),\ x \neq y, \tag{5}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order one. It is well known that there exists a unique solution u^{sca} of the problem (2)–(3), and it has the following asymptotic behaviour,

$$u^{sca}(x, \theta) = \frac{\mathrm{e}^{ikr}}{\sqrt{r}}\Big\{u^\infty(\hat{x}, \theta) + O(1/r)\Big\},\ r \to \infty, \tag{6}$$

where $\hat{x} := \frac{x}{|x|}$. The function u^∞ is called the *far field pattern* of u^{sca}, and it has the form

$$u^\infty(\hat{x}, \theta) = \frac{k^2 \mathrm{e}^{\frac{i\pi}{4}}}{\sqrt{8\pi k}} \int_Q \mathrm{e}^{-ik\hat{x}\cdot y} u(y, \theta)q(y)dy =: \mathcal{F}_\theta q(\hat{x}), \tag{7}$$

where the far field mapping $\mathcal{F}_\theta : L^2(Q) \to L^2(\mathbb{S}^1)$ defined in the second equality is nonlinear. We consider the inverse scattering problem to reconstruct the function q from the far field pattern $u^\infty(\cdot, \theta_n)$ with several directions $\{\theta_n\}_{n=1}^N \subset \mathbb{S}^1$, that is, to solve the following nonlinear problem with respect to q:

$$\mathcal{F}_{\theta_n} q = u^\infty_{\theta_n},\ n = 1, ..., N. \tag{8}$$

For more details on direct and inverse scattering problems, see Chapter 8 of [1].

2. Kalman filter

We review the Kalman filter in a general functional analytic setting. Let X and Y be Hilbert spaces over complex variables $\mathbb{C}$, $f_n \in Y$ $(n = 1, ..., N)$ be a measurement, and $A_n : X \to Y$ $(n = 1, ..., N)$ be a linear operator from X to Y. We consider the problem of determining $\varphi \in X$ such that

$$A_n \varphi = f_n, \tag{9}$$

for all $n = 1, ..., N$. Now, we assume that we have the initial guess $\varphi_0 \in X$, which is the starting point of the algorithm, and that it was appropriately determined by a priori information of the true solution φ^{true}. Then, we consider the minimization problem of the following functional.

$$\begin{aligned} J_{Full,N}(\varphi) &:= \alpha \left\|\varphi - \varphi_0\right\|_X^2 + \left\|\vec{f} - \vec{A}\varphi\right\|_{Y^N,R^{-1}}^2 \\ &= \alpha \left\|\varphi - \varphi_0\right\|_X^2 + \sum_{n=1}^{N} \left\|f_n - A_n\varphi\right\|_{Y,R^{-1}}^2, \end{aligned} \tag{10}$$

where $\vec{f} := \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$ and $\vec{A} := \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix}$. The norm $\|\cdot\|_{Y,R^{-1}}^2 := \langle \cdot, R^{-1} \cdot \rangle_Y$ is a weighted norm with a positive definite symmetric invertible operator $R : Y \to Y$. The minimizer of (10) is given by

$$\varphi_N^{FT} := \varphi_0 + (\alpha I + \vec{A}^*\vec{A})^{-1}\vec{A}^*\left(\vec{f} - \vec{A}\varphi_0\right). \tag{11}$$

We call this the *Full data Tikhonov*. Here, $\vec{A}^*$ is the adjoint operator with respect to $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_{Y^N,R^{-1}}$. We calculate

$$\begin{aligned} \langle \vec{f}, \vec{A}\varphi\rangle_{Y^N,R^{-1}} &= \sum_{n=1}^{N} \langle f_n, R^{-1}A_n\varphi\rangle_Y \\ &= \sum_{n=1}^{N} \langle A_n^H R^{-1} f_n, \varphi\rangle_X = \langle \vec{A}^H R^{-1}\vec{f}, \varphi\rangle_X, \end{aligned} \tag{12}$$

which implies that

$$\vec{A}^* = \vec{A}^H R^{-1}, \tag{13}$$

where A_n^H and $\vec{A}^H$ are the adjoint operators with respect to the usual scalar products $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_Y$ and $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_{Y^N}$, respectively. Then, the Full data Tikhonov solution in (11) is of the form

$$\varphi_N^{FT} = \varphi_0 + \left(\alpha I + \vec{A}^H R^{-1}\vec{A}\right)^{-1}\vec{A}^H R^{-1}\left(\vec{f} - \vec{A}\varphi_0\right). \tag{14}$$

However, algorithm (14) of the Full data Tikhonov is computationally expensive because we must construct a larger vector $\vec{f}$ and a large operator $\vec{A}$ when the number of measurements N increases. Accordingly, we consider the alternative approach based on the Kalman filter (see, e.g., [2,6]), which is the algorithm give by the following algorithm:

$$\varphi_n^{KF} := \varphi_{n-1}^{KF} + K_n\left(f_n - A_n\varphi_{n-1}^{KF}\right), \tag{15}$$

$$K_n := B_{n-1}A_n^H\left(R + A_nB_{n-1}A_n^H\right)^{-1}, \tag{16}$$

$$B_n := \left(I - K_nA_n\right)B_{n-1}, \tag{17}$$

for $n = 1, ..., N$, where $\varphi_0^{KF} := \varphi_0$ and $B_0 := \frac{1}{\alpha}I$. Here, φ_n^{KF} is the unique minimizer of the following functional (see Section 7 of [2] and Section 5 of [6]):

$$J_{KF,n}(\varphi) := \left\|\varphi - \varphi_{n-1}^{KF}\right\|_{X,B_{n-1}^{-1}}^2 + \left\|f_n - A_n\varphi\right\|_{Y,R^{-1}}^2. \tag{18}$$

We observe that the Kalman filter algorithm updates state φ every n with measurement f_n and one operator A_n, and it does not require large vectors or operators. Instead, it updates both the state φ in (15) and weight B of the norm in (17), which plays the role of retaining the information from past updates. By the same argument in Theorem 5.4.7 of [6], we can prove the equivalence of the Full data Tikhonov and Kalman filter when all observation operators A_n are linear.

Lemma 1. *For measurements $f_1, ..., f_N$, linear operators $A_1, ..., A_N$, and the initial guess $\varphi_0 \in X$, the final state of the Kalman filter given by (15)–(17) is equivalent to the state of the Full data Tikhonov given by (14), that is,*

$$\varphi_N^{KF} = \varphi_N^{FT}. \tag{19}$$

3. Kalman filter Levenberg–Marquardt

In this section, we propose a reconstruction algorithm based on the Kalman filter that is equivalent to the Levenberg–Marquardt algorithm. We solve the following problem with respect to q:

$$\mathcal{F}_{\theta_n}q = u_{\theta_n}^{\infty},\ n = 1, ..., N. \tag{20}$$

It is convenient to employ the vector notation as follows:

$$\vec{\mathcal{F}}q = \vec{u}^{\infty}, \tag{21}$$

where $\vec{\mathcal{F}}q := \begin{pmatrix} \mathcal{F}_{\theta_1}q \\ \vdots \\ \mathcal{F}_{\theta_N}q \end{pmatrix}$ and $\vec{u}^{\infty} := \begin{pmatrix} u_{\theta_1}^{\infty} \\ \vdots \\ u_{\theta_N}^{\infty} \end{pmatrix}$.

First, we review the derivation of the Levenberg-Marquardt scheme (see, e.g., [5]) as follows. We assume that we have an initial guess q_0 and consider the Taylor expansion at $q = q_0$.

$$\vec{\mathcal{F}}q = \vec{\mathcal{F}}q_0 + \vec{\mathcal{F}}'[q_0](q - q_0) + r(q - q_0). \tag{22}$$

We forget the high-order term $r(q - q_0)$ and solve the linearized problem for (21):

$$\vec{\mathcal{F}}'[q_0]q = \vec{u}^{\infty} - \vec{\mathcal{F}}q_0 + \vec{\mathcal{F}}'[q_0]q_0, \tag{23}$$

where $\vec{\mathcal{F}}'[q_0]q := \begin{pmatrix} \mathcal{F}_{\theta_1}'[q_0]q \\ \vdots \\ \mathcal{F}_{\theta_N}'[q_0]q \end{pmatrix}$. Then, the Tikhonov regularization solution is given by

$$q_1 := q_0 + \left(\alpha_0 I + \vec{\mathcal{F}}'[q_0]^*\vec{\mathcal{F}}'[q_0]\right)^{-1}\vec{\mathcal{F}}'[q_0]^*\left(\vec{u}^{\infty} - \vec{\mathcal{F}}q_0\right), \tag{24}$$

where $\alpha_0 > 0$ is a regularization parameter.

Next, we solve the linearized problem for (21) with initial guess q_1:

$$\vec{\mathcal{F}}'[q_1]q = \vec{u}^{\infty} - \vec{\mathcal{F}}q_1 + \vec{\mathcal{F}}'[q_1]q_1. \tag{25}$$

Then, the Tikhonov regularization solution is given by

$$q_2 := q_1 + \left(\alpha_1 I + \vec{\mathcal{F}}'[q_1]^* \vec{\mathcal{F}}'[q_1]\right)^{-1} \vec{\mathcal{F}}'[q_1]^* \left(\vec{u}^{\infty} - \vec{\mathcal{F}}q_1\right), \tag{26}$$

where $\alpha_1 > 0$ is a regularization parameter. Repeating the above arguments (22)–(26), we have the iteration scheme for $i \in \mathbb{N}_0$:

$$\begin{aligned} q_{i+1}^{FLM} := q_i^{FLM} &+ \left(\alpha_i I + \vec{\mathcal{F}}'[q_i^{FLM}]^* \vec{\mathcal{F}}'[q_i^{FLM}]\right)^{-1} \\ &\times \vec{\mathcal{F}}'[q_i^{FLM}]^* \left(\vec{u}^{\infty} - \vec{\mathcal{F}}q_i^{FLM}\right), \end{aligned} \tag{27}$$

where $\{\alpha_i\}_{i\in\mathbb{N}_0}$ is a sequence of regularization parameters. We call this the *Full data Levenberg–Marquardt* (FLM). Here, $\vec{\mathcal{F}}'[q_i^{FLM}]^*$ is an adjoint operator of $\vec{\mathcal{F}}'[q_i^{FLM}]$ with respect to the usual scalar product $\langle\cdot,\cdot\rangle_{L^2(Q)}$ and weighted scalar product $\langle\cdot,\cdot\rangle_{L^2(\mathbb{S}^1)^N, R^{-1}}$, where $R : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is the positive definite symmetric invertible linear operator. By the same calculation as in (12), we have

$$\vec{\mathcal{F}}'[q_i^{FLM}]^* = \vec{\mathcal{F}}'[q_i^{FLM}]^H R^{-1}, \tag{28}$$

where $\vec{\mathcal{F}}'[q_i^{LM}]^H$ is an adjoint operator of $\vec{\mathcal{F}}'[q_i^{LM}]$ with respect to usual scalar products $\langle\cdot,\cdot\rangle_{L^2(Q)}$ and $\langle\cdot,\cdot\rangle_{L^2(\mathbb{S}^1)^N}$. Then, (27) can be of the form

$$\begin{aligned} q_{i+1}^{FLM} = q_i^{FLM} &+ \left(\alpha_i I + \vec{\mathcal{F}}'[q_i^{LM}]^H R^{-1} \vec{\mathcal{F}}'[q_i^{LM}]\right)^{-1} \\ &\times \vec{\mathcal{F}}'[q_i^{LM}]^H R^{-1} \left(\vec{u}^{\infty} - \vec{\mathcal{F}}q_i^{FLM}\right). \end{aligned} \tag{29}$$

As stated in Section 2., algorithm (29) is computationally expensive when the number of measurements N increases. Accordingly, we consider the alternative approach based on the Kalman filter. We denote

$$q_{0,0} := q_0 \ \text{ and } \ B_{0,0} := \frac{1}{\alpha_0} I. \tag{30}$$

We solve the linearized problem for (20) with initial guess $q_{0,0}$:

$$\mathcal{F}'_{\theta_n}[q_{0,0}]q = u^{\infty}_{\theta_n} - \mathcal{F}_{\theta_n} q_{0,0} + \mathcal{F}'_{\theta_n}[q_{0,0}]q_{0,0}, \tag{31}$$

for $n = 1, ..., N$. Applying the Kalman filter update (15)–(17) as

$$f_n = u^{\infty}_{\theta_n} - \mathcal{F}_{\theta_n} q_{0,0} + \mathcal{F}'_{\theta_n}[q_{0,0}]q_{0,0}, \ \ A_n = \mathcal{F}'_{\theta_n}[q_{0,0}],$$

$$\varphi_0 = q_{0,0}, \ \text{ and } \ B_0 = B_{0,0}, \tag{32}$$

we obtain the algorithm for $n = 1, ..., N$,

$$\begin{aligned} q_{0,n} &:= q_{0,n-1} + K_{0,n} \\ &\times \left(u^{\infty}_{\theta_n} - \mathcal{F}_{\theta_n} q_{0,0} + \mathcal{F}'_{\theta_n}[q_{0,0}]q_{0,0} - \mathcal{F}_{\theta_n}[q_{0,0}]q_{0,n-1}\right), \end{aligned} \tag{33}$$

$$\begin{aligned} K_{0,n} :=& B_{0,n-1}\mathcal{F}'_{\theta_n}[q_{0,0}]^H \\ &\times \left(R + \mathcal{F}'_{\theta_n}[q_{0,0}] B_{0,n-1} \mathcal{F}'_{\theta_n}[q_{0,0}]^H\right)^{-1}, \end{aligned} \tag{34}$$

$$B_{0,n} := \left(I - K_{0,n}\mathcal{F}'_{\theta_n}[q_{0,0}]\right) B_{0,n-1}. \tag{35}$$

Next, we denote

$$q_{1,0} := q_{0,N} \ \text{ and } \ B_{1,0} := \frac{1}{\alpha_1} I, \tag{36}$$

We solve the linearized problem for (20) with initial guess $q_{1,0}$:

$$\mathcal{F}'_{\theta_n}[q_{1,0}]q = u^{\infty}_{\theta_n} - \mathcal{F}_{\theta_n} q_{1,0} + \mathcal{F}'_{\theta_n}[q_{1,0}]q_{1,0}, \tag{37}$$

for $n = 1, ..., N$. Applying the Kalman filter update (15)–(17) as

$$f_n = u^{\infty}_{\theta_n} - \mathcal{F}_{\theta_n} q_{1,0} + \mathcal{F}'_{\theta_n}[q_{1,0}]q_{1,0}, \ \ A_n = \mathcal{F}'_{\theta_n}[q_{1,0}],$$

$$\varphi_0 = q_{1,0}, \ \text{ and } \ B_0 = B_{1,0}, \tag{38}$$

we obtain the algorithm for $n = 1, ..., N$:

$$\begin{aligned} q_{1,n} &:= q_{1,n-1} + K_{1,n} \\ &\times \left(u^{\infty}_{\theta_n} - \mathcal{F}_{\theta_n} q_{1,0} + \mathcal{F}'_{\theta_n}[q_{1,0}]q_{1,0} - \mathcal{F}_{\theta_n}[q_{1,0}]q_{1,n-1}\right), \end{aligned} \tag{39}$$

$$\begin{aligned} K_{1,n} :=& B_{1,n-1}\mathcal{F}'_{\theta_n}[q_{1,0}]^H \\ &\times \left(R + \mathcal{F}'_{\theta_n}[q_{1,0}] B_{1,n-1} \mathcal{F}'_{\theta_n}[q_{1,0}]^H\right)^{-1}, \end{aligned} \tag{40}$$

$$B_{1,n} := \left(I - K_{1,n}\mathcal{F}'_{\theta_n}[q_{1,0}]\right) B_{1,n-1}. \tag{41}$$

Repeating the above arguments (30)–(41), we obtain the following algorithm for $n = 1, ..., N$ and $i \in \mathbb{N}_0$:

$$\begin{aligned} q_{i,n}^{KFL} &:= q_{i,n-1}^{KFL} + K_{i,n}\Big(u^{\infty}_{\theta_n} - \mathcal{F}_{\theta_n} q_{i,0}^{KFL} \\ &+ \mathcal{F}'_{\theta_n}[q_{i,0}^{KFL}]q_{i,0}^{KFL} - \mathcal{F}_{\theta_n}[q_{i,0}^{KFL}]q_{i,n-1}^{KFL}\Big), \end{aligned} \tag{42}$$

$$\begin{aligned} K_{i,n} :=& B_{i,n-1}\mathcal{F}'_{\theta_n}[q_{i,0}^{KFL}]^H \\ &\times \left(R + \mathcal{F}'_{\theta_n}[q_{i,0}^{KFL}] B_{i,n-1} \mathcal{F}'_{\theta_n}[q_{i,0}^{KFL}]^H\right)^{-1}, \end{aligned} \tag{43}$$

$$B_{i,n} := \left(I - K_{i,n}\mathcal{F}'_{\theta_n}[q_{i,0}^{KFL}]\right) B_{i,n-1}, \tag{44}$$

where

$$q_{i,0}^{KFL} := q_{i-1,N}^{KFL}, \tag{45}$$

$$B_{i,0} := \frac{1}{\alpha_i} I. \tag{46}$$

We call this the *Kalman filter Levenberg–Marquardt* (KFL). We remark that the algorithm has indexes i and n, where i is associated with the iteration step and n with the measurement step, respectively. It is shown that in [3], the KFL is equivalent to the FLM.

Theorem 2. *For the initial guess $q_0 \in L^2(Q)$ and sequence $\{\alpha_i\}_{i\in\mathbb{N}_0}$ of the regularization parameters, the Kalman filter Levenberg–Marquardt (42)–(46) is equivalent to the Full data Levenberg–Marquardt given by (29), that is, for all $i \in \mathbb{N}_0$, we have*

$$q_{i,N}^{KFL} = q_{i+1}^{FLM}. \tag{47}$$

Figures 1 and 2 illustrates the FLM and KFL, respectively. While the FLM only moves horizontally, the KFL first moves vertically. Once it was moved up to $n = N$, it moves horizontally, and then the linearization is complete.

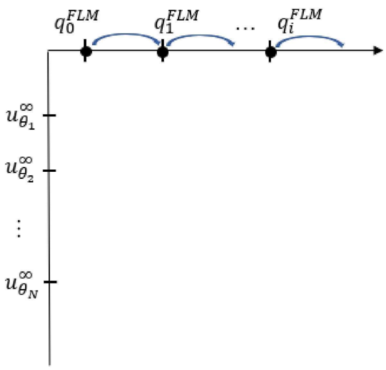

Fig. 1 Illustration of FLM

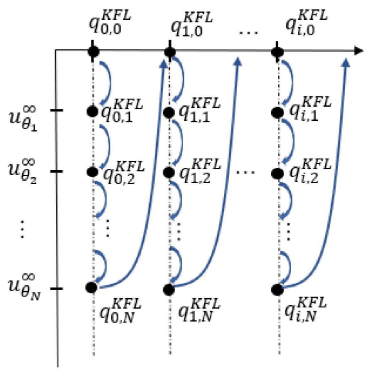

Fig. 2 Illustration of KFL

4. Iterative Extended Kalman filter

In this section, we propose the algorithm inspired by the Extended Kalman filter (see, e.g., [4]). We denote

$$q_{0,0} := q_0 \quad \text{and} \quad B_{0,0} := \frac{1}{\alpha_0} I. \tag{48}$$

We solve the linearized problem for the equation

$$u_{\theta_1}^\infty = \mathcal{F}_{\theta_1} q, \tag{49}$$

with respect to $n = 1$ and initial guess $q_{0,0}$, that is,

$$\mathcal{F}'_{\theta_1}[q_{0,0}]q = u_{\theta_1}^\infty - \mathcal{F}_{\theta_1} q_{0,0} + \mathcal{F}'_{\theta_1}[q_{0,0}]q_{0,0}, \tag{50}$$

which is equivalent to solving the minimization problem of the following functional:

$$\begin{aligned} J_{0,0}(q) &:= \|q - q_{0,0}\|^2_{X,B_{0,0}^{-1}} \\ &+ \left\| u_{\theta_1}^\infty - \mathcal{F}_{\theta_1} q_{0,0} + \mathcal{F}'_{\theta_1}[q_{0,0}]q_{0,0} - \mathcal{F}'_{\theta_1}[q_{0,0}]q \right\|^2_{Y,R^{-1}}. \end{aligned} \tag{51}$$

By the same argument as in Section 7 of [2], the Tikhonov regularization solution has the following form:

$$q_{0,1} := q_{0,0} + K_{0,1}\left(u_{\theta_1}^\infty - \mathcal{F}_{\theta_1} q_{0,0}\right), \tag{52}$$

$$K_{0,1} := B_{0,0}\mathcal{F}'_{\theta_1}[q_{0,0}]^H \left(R + \mathcal{F}'_{\theta_1}[q_{0,0}]B_{0,0}\mathcal{F}'_{\theta_1}[q_{0,0}]^H\right)^{-1}, \tag{53}$$

$$B_{0,1} := \left(I - K_{0,1}\mathcal{F}'_{\theta_1}[q_{0,0}]\right) B_{0,0}. \tag{54}$$

Next, we solve the linearized problem for the equation

$$u_{\theta_2}^\infty = \mathcal{F}_{\theta_2} q, \tag{55}$$

with respect to $n = 2$ and initial guess $q_{0,1}$, that is,

$$\mathcal{F}'_{\theta_2}[q_{0,1}]q = u_{\theta_2}^\infty - \mathcal{F}_{\theta_2} q_{0,1} + \mathcal{F}'_{\theta_2}[q_{0,1}]q_{0,1}, \tag{56}$$

which is equivalent to solving the minimization problem of the following functional:

$$\begin{aligned} J_{0,1}(q) &:= \|q - q_{0,1}\|^2_{X,B_{0,1}^{-1}} \\ &+ \left\| u_{\theta_2}^\infty - \mathcal{F}_{\theta_2} q_{0,1} + \mathcal{F}'_{\theta_2}[q_{0,1}]q_{0,1} - \mathcal{F}'_{\theta_2}[q_{0,1}]q \right\|^2_{Y,R^{-1}}. \end{aligned} \tag{57}$$

The Tikhonov regularization solution has the following form:

$$q_{0,2} := q_{0,1} + K_{0,2}\left(u_{\theta_2}^\infty - \mathcal{F}_{\theta_2} q_{0,1}\right), \tag{58}$$

$$K_{0,2} := B_{0,1}\mathcal{F}'_{\theta_2}[q_{0,1}]^H \left(R + \mathcal{F}'_{\theta_2}[q_{0,1}]B_{0,1}\mathcal{F}'_{\theta_2}[q_{0,1}]^H\right)^{-1}, \tag{59}$$

$$B_{0,2} := \left(I - K_{0,2}\mathcal{F}'_{\theta_2}[q_{0,1}]\right) B_{0,1}. \tag{60}$$

Repeating the above arguments (48)–(60), we obtain the algorithm for $n = 1, ..., N$:

$$q_{0,n} := q_{0,n-1} + K_{0,n-1}\left(u_{\theta_n}^\infty - \mathcal{F}_{\theta_n} q_{n-1}\right), \tag{61}$$

$$\begin{aligned} K_{0,n} &:= B_{0,n-1}\mathcal{F}'_{\theta_n}[q_{0,n-1}]^H \\ &\times \left(R + \mathcal{F}'_{\theta_n}[q_{0,n-1}]B_{0,n-1}\mathcal{F}'_{\theta_n}[q_{0,n-1}]^H\right)^{-1}, \end{aligned} \tag{62}$$

$$B_{0,n} := \left(I - K_{0,n}\mathcal{F}'_{\theta_n}[q_{0,n-1}]\right) B_{0,n-1}. \tag{63}$$

Next, we denote

$$q_{1,0} := q_{0,N}, \quad \text{and} \quad B_{1,0} := \frac{1}{\alpha_1} I. \tag{64}$$

We solve the linearized problem for $u_{\theta_1}^\infty = \mathcal{F}_{\theta_1} q$ with respect to $n = 1$ and initial guess $q_{1,0}$, that is,

$$\mathcal{F}'_{\theta_1}[q_{1,0}]q = u_{\theta_1}^\infty - \mathcal{F}_{\theta_1} q_{1,0} + \mathcal{F}'_{\theta_1}[q_{1,0}]q_{1,0}. \tag{65}$$

The Tikhonov regularization solution has the following form:

$$q_{1,1} := q_{1,0} + K_1 \left(u^{\infty}_{\theta_1} - \mathcal{F}_{\theta_1} q_{1,0} \right), \tag{66}$$

$$K_{1,1} := B_{1,0} \mathcal{F}'_{\theta_1}[q_{1,0}]^H \left(R + \mathcal{F}'_{\theta_1}[q_{1,0}] B_{1,0} \mathcal{F}'_{\theta_1}[q_{1,0}]^H \right)^{-1}, \tag{67}$$

$$B_{1,1} := \left(I - K_{1,1} \mathcal{F}'_{\theta_1}[q_{1,0}] \right) B_{1,0}. \tag{68}$$

By solving the linearized problem for $u^{\infty}_{\theta_n} = \mathcal{F}_{\theta_n} q$ up to $n = N$, we obtain $q_{1,N}$, $B_{1,N}$, and denote $q_{2,0} := q_{1,N}$, $B_{2,0} := \frac{1}{\alpha_2} I$,. Repeating the above arguments, we finally obtain the following algorithm for $n = 1, ..., N$ and $i \in \mathbb{N}_0$:

$$q^{EKF}_{i,n} := q^{EKF}_{i,n-1} + K_{i,n-1} \left(u^{\infty}_{\theta_n} - \mathcal{F}_{\theta_n} q^{EKF}_{i,n-1} \right), \tag{69}$$

$$\begin{aligned} K_{i,n} &:= B_{i,n-1} \mathcal{F}'_{\theta_n}[q^{EKF}_{i,n-1}]^H \\ &\times \left(R + \mathcal{F}'_{\theta_n}[q^{EKF}_{i,n-1}] B_{i,n-1} \mathcal{F}'_{\theta_n}[q^{EKF}_{i,n-1}]^H \right)^{-1}, \end{aligned} \tag{70}$$

$$B_{i,n} := \left(I - K_{i,n} \mathcal{F}'_{\theta_n}[q^{EKF}_{i,n-1}] \right) B_{i,n-1}, \tag{71}$$

where

$$q^{EKF}_{i,0} := q^{EKF}_{i-1,N}, \tag{72}$$

$$B_{i,0} := \frac{1}{\alpha_i} I. \tag{73}$$

We call this the *iterative Extended Kalman filter* (EKF). As remarked in Section 3., the algorithm has indexes i and n, where i is associated with the iteration step and n with the measurement step, respectively. Figure 3 illustrates the EKF. EKF always moves diagonally because linearization is performed in every measurement step.

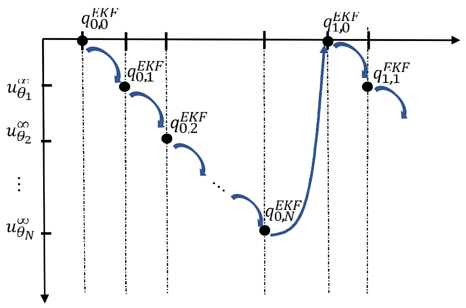

Fig. 3 Illustration of EKF

Remark 3. We compare the KFL with the EKF. KFL is based on the linearization at the initial state for each iteration step, whereas EKF is based on the linearization at the current state for every iteration step, implying that the update of the KFL is slower than that of the EKF.

Remark 4. In both the KFL and EKF algorithms, instead of initializations (46) and (73), we can update the weight of the norm for each iteration step, that is,

$$B_{i,0} := B_{i-1,N}, \tag{74}$$

which plays a role in retaining the information of past updates as iteration step i proceeds.

5. Numerical examples

In this section, we provide numerical examples to demonstrate the algorithms. The inverse scattering problem concerns solving the nonlinear integral equation for $n = 1, ..., N$

$$\mathcal{F}_{\theta_n} q = u^{\infty}(\cdot, \theta_n), \tag{75}$$

where the operator $\mathcal{F}_{\theta_n} : L^2(Q) \to L^{\infty}(\mathbb{S}^1)$ is defined by

$$\mathcal{F}_{\theta_n} q(\hat{x}) = \frac{k^2 \mathrm{e}^{\frac{i\pi}{4}}}{\sqrt{8\pi k}} \int_Q \mathrm{e}^{-ik\hat{x}\cdot y} u_q(y, \theta_n) q(y) dy, \tag{76}$$

where the incident direction is denoted by $\theta_n := (\cos(2\pi n/N), \sin(2\pi n/N))$. Here, $u_q(\cdot, \theta_n)$ is the solution of the Lippmann-Schwinger integral equation (4), which is numerically calculated based on Vainikko's method [7]. This is a fast solution method for the Lippmann–Schwinger equation based on periodization, fast Fourier transform techniques, and multi-grid methods. We assume that the support of function q is included in $[-S, S]^2$ with some $S > 0$, and function q is discretized by a piecewise constant on $[-S, S]^2$ decomposed by squares with length $\frac{S}{M}$, that is,

$$q \approx \left(q(y_{m_1, m_2}) \right)_{-M \le m_1, m_2 \le M-1} \in \mathbb{C}^{(2M)^2}, \tag{77}$$

where $y_{m_1,m_2} := \left(\frac{(2m_1+1)S}{2M}, \frac{(2m_2+1)S}{2M} \right)$, and $M \in \mathbb{N}$ is a number of the division of $[0, S]$. Furthermore, the function $u^{\infty}(\cdot, \theta_n)$ is discretized by

$$u^{\infty}_n(\cdot, \theta_n) \approx \left(u^{\infty}(\hat{x}_j, \theta_n) \right)_{j=1,...,J} \in \mathbb{C}^J, \tag{78}$$

where $\hat{x}_j := (\cos(2\pi j/J), \sin(2\pi j/J))$, and $J \in \mathbb{N}$ is a number of the division of $[0, 2\pi]$.

We always fix the following parameters as $J = 60$, $M = 6$, $S = 3$, $N = 60$, and $k = 7$. We consider true function as the characteristic function

$$q^{true}(x) := \begin{cases} 1.0 & \text{for } x \in B \\ 0 & \text{for } x \notin B \end{cases}, \tag{79}$$

where the support B of the true function is considered as follows:

$$B := \left\{ (x_1, x_2) : x_1^2 + x_2^2 < 1.0 \right\}, \tag{80}$$

In Figure 4, the closed blue curve is the boundary ∂B of the support B, and the green brightness indicates the value of the true function on each cell divided into $(2M)^2$ in the sampling domain $[-S, S]^2$. Here, we always employ the initial guess q_0 as

$$q_0 \equiv 0. \tag{81}$$

We demonstrate four algorithms: the Extended Kalman filter (69)–(73) with initialization (73) (EKF-initialization), Extended Kalman filter (69)–(73) with update (74) (EKF-update), Kalman filter Levenberg–Marquardt (42)–(44) with initialization (46) (KFL-initialization), and Kalman filter

Levenberg–Marquardt (42)–(44) with update (74) (KFL-initialization). Figures 5 show the reconstruction by the four algorithms. The first and second rows correspond to a visualization of the four algorithms, and the third row is the graph of the Mean Square Error (MSE) defined by

$$e_i := \left\| q^{true} - q_i \right\|^2 , \qquad (82)$$

where q_i is associated with the state of the i-th iteration step. The horizontal and vertical axes correspond to the number of iterations and the MSE value, respectively.

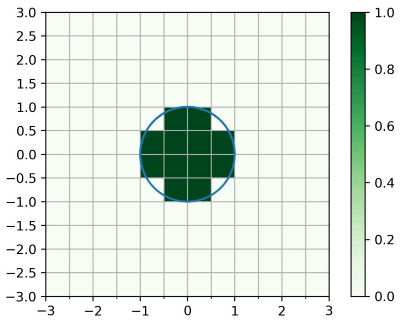

Fig. 4 True function q^{true}

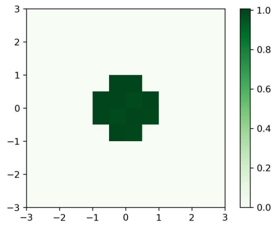

Fig. 5-a EKF-initialization

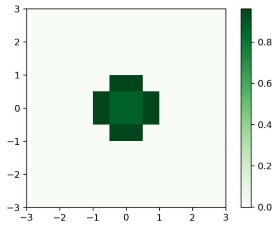

Fig. 5-b KFL-initialization

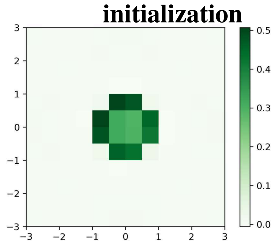

Fig. 5-c EKF-update

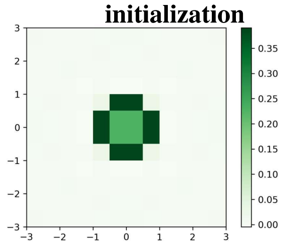

Fig. 5-d KFL-update

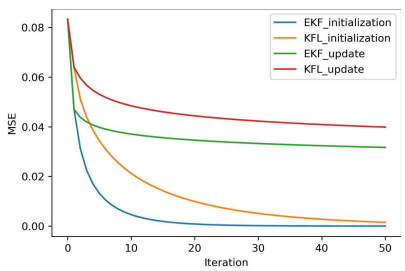

Fig. 5 Reconstruction

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